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On the Effective Action of Confining Strings

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Abstract: We study the low-energy effective action on confining strings (in the fundamental representation) in $SU(N)$ gauge theories in D space-time dimensions. We write this action in terms of the physical transverse fluctuations of the string. We show that for any D, the four-derivative terms in the effective action must exactly match the ones in the Nambu-Goto action, generalizing a result of Lüscher and Weisz for $D = 3$. We then analyze the six-derivative terms, and we show that some of these terms are constrained. For $D = 3$ this uniquely determines the effective action for closed strings to this order, while for $D > 3$ one term is not uniquely determined by our considerations. This implies that for $D = 3$ the energy levels of a closed string of length L agree with the Nambu-Goto result at least up to order $1/L^5$. For any D we find that the partition function of a long string on a torus is unaffected by the free coefficient, so it is always equal to the Nambu-Goto partition function up to six-derivative order. For a closed string of length L, this means that for $D > 3$ its energy can, in principle, deviate from the Nambu-Goto result at order $1/L^5$, but such deviations must always cancel in the computation of the partition function (so that the sum of the deviations of all states at each energy level must vanish). In particular there is no correction at this order to the ground state energy of a winding string. Next, we compute the effective action up to six-derivative order for the special case of confining strings in weakly-curved holographic backgrounds, at one-loop order (leading order in the curvature). Our computation is general, and applies in particular to backgrounds like the Witten background, the Maldacena-Nuñez background, and the Klebanov-Strassler background. We show that this effective action obeys all of the constraints we derive, and in fact it precisely agrees with the Nambu-Goto action (the single allowed deviation does not appear).

Contents

1. Introduction

The confining string is a basic object in confining $SU(N)$ gauge theories, in particular when there is no matter in representations of non-zero N-ality, such that this string is stable. Like any other solitonic object, it is interesting to study the low-energy effective action on this string (at energies much lower than the QCD scale), in order to understand its low-energy fluctuations and the light excitations of long strings. This study is particularly interesting since the confining string in large N gauge theories is believed to be a weakly coupled fundamental string moving in some background (with a string coupling of order $1/N$ [1]).

When this background is known, we can use it to compute the low-energy effective action on the string. For most interesting confining theories the corresponding string background is not yet known, and one can hope that studying (say, by lattice simulations) the low-energy effective action on a confining string could teach us about the properties of this background, and give clues for its construction.

The simplest effective action for a confining string in D space-time dimensions is the Nambu-Goto action, which is simply the string tension T times the area of the string worldsheet. A priori there is no reason why the effective action on confining strings should take this simple form, but lattice simulations for pure Yang-Mills theories in $D = 3$ and $D = 4$ show (as we will review below) that the effective action is very well approximated by the Nambu-Goto form, with only very small deviations. Our goal in this paper will be to understand why this is the case, and to estimate at what order deviations from the Nambu-Goto action are expected to occur.

Two main approaches to constrain the effective action of a confining string have been studied in the literature. The Polchinski-Strominger approach [2, 3, 4] uses a conformal gauge on the worldsheet, in which the degrees of freedom in the effective action are the D embedding coordinates of the string. In this approach the constraints on the effective action come by requiring that it must have the correct (critical) central charge, and it was shown that this implies that the four-derivative effective action must agree with the Nambu-Goto form. However, it seems difficult to extend this approach to higher orders. In this paper we follow the second approach [5, 6, 7], writing the effective action in static gauge, such that the degrees of freedom are only the $(D - 2)$ transverse fluctuations of the string worldsheet. Lüscher and Weisz argued in [7] that by computing the partition function of long winding strings, and expressing it as a sum over string states, one can constrain the effective action; they showed that the partition function on the annulus constrains the four-derivative terms in $D = 3$ to be of Nambu-Goto form, but that for $D > 3$ there seems to be one undetermined coefficient. Essentially, the information that goes into this approach is [8] that the action should non-linearly realize the Lorentz symmetry rotating the direction that the string propagates in and the transverse directions.

In the first part of this paper we generalize the approach of Lüscher and Weisz in two directions. First, we compute the four-derivative partition function of a long winding string on the torus, and we show that this constrains the four-derivative terms in the effective action to be of Nambu-Goto form for any D. Then, we extend the computation to the six-derivative terms, computing the partition function on the torus and on the annulus. For general D we show that the considerations of Lüscher and Weisz allow us to determine two of the three free coefficients at six-derivative order, but that one coefficient remains unfixed. Strangely, it turns out that this free coefficient does not affect the partition function of the long string on the torus, so that if there are corrections to energy levels at six-derivative order (order $1/L^5$ for a string of length L) they must cancel exactly in the partition function. In particular, our results imply that the ground state energy of a closed winding string is exactly given by its Nambu-Goto form up to order $1/L^5$, and can deviate from this form only starting at order $1/L^7$. For $D = 3$ we show that the effective action is uniquely determined up to six-derivative order, so that the previous sentence

applies to all states of closed winding strings. The computations of the partition functions require a regularization of the effective action, and we use (following [9, 7]) a zeta function regularization, which gives finite results, independent of the UV cutoff.

In most confining gauge theories we do not know how to compute directly the effective action on the confining string, and we can only do it numerically by lattice simulations. However, in the past decade a new class of confining gauge theories was discovered, whose dual string theory description lives in a weakly curved background. In such backgrounds we can compute explicitly the effective action on the confining string, and we perform this computation to six-derivative order in the second part of our paper. More precisely, we compute the leading dependence of the terms in the six-derivative effective action on the curvature of the background (which typically maps to some negative power of the 't Hooft coupling). There are several motivations for this computation :

- It is the first example (as far as we know) of a direct computation of the effective action on a confining string.
- We show that the effective action we compute obeys all the constraints discussed in the previous paragraph, thus enabling us to test both the form of these constraints and our computation of the effective action.
- Our computation allows us to check whether the term in the effective action that is allowed to deviate from the Nambu-Goto form is actually present or not. This is important since there may be additional constraints on the effective action that may set this term to zero. We find that, at the leading order that we work in, there are actually no deviations from the Nambu-Goto action.
- Some of the backgrounds we study are continuously related (by changing a dimensionless parameter) to pure Yang-Mills and pure super Yang-Mills theories in $D = 3$ and $D = 4$, and we expect that the qualitative form of the effective action will not change when we change the parameters.

We begin in section 2 with general comments on the effective action on confining strings, and with a review of the known results. In section 3 we generalize the computations of [7] to the torus partition function and to the next order in the derivative expansion. In section 4 we write down the worldsheet action for strings in weakly curved confining backgrounds, and the Feynman rules that follow from it. Our discussion in this section is general, and in the following section 5 we discuss in detail some of the examples to which our considerations apply. In section 6 we use this worldsheet action to compute the effective action on the corresponding confining strings, at leading order in the space-time curvature. We end in section 7 with our conclusions. Two appendices contain some technical details. In appendix A we present the computations used in section 3, and in appendix B we review our conventions for sections 4-6.

2. General features of the effective action of a confining string

2.1 Generalities

In this paper we consider confining gauge theories in which the confining string is absolutely stable. For $SU(N)$ gauge theories, this means that there cannot be any dynamical fields in representations with non-zero N-ality, such as the fundamental representation. Of course, in the large N limit the confining string becomes stable even in the presence of dynamical fields in the fundamental representation. For finite N, in the presence of such fields, the string can break.

In this situation it makes sense to ask about the low-energy effective action on a long, straight confining string. A string-like object in a D-dimensional gauge theory breaks $(D-2)$ translation symmetries, so there should be $(D-2)$ massless Nambu-Goldstone bosons on the worldsheet, whose expectation values are simply the transverse positions of the string. In a generic confining theory we do not expect any additional massless fields on the string worldsheet, so the effective action will involve only these massless fields. In theories with additional symmetries there may be additional massless fields on the worldsheet. For instance, in supersymmetric gauge theories, the confining string typically breaks all the supersymmetry, so it should have additional massless fermions on its worldvolume; for instance the confining string in the $D=4$ $\mathcal{N}=1$ supersymmetric Yang-Mills theory should have 4 massless Majorana-Weyl fermions on its worldvolume. In this paper we will ignore the possibility of having such additional fields, though we expect that they will not change most of our conclusions. It would be interesting to generalize our analysis to include additional massless fields arising from additional symmetries.

The effective theory of the Nambu-Goldstone bosons is independent of their expectation value, so all interactions involve derivatives of all the fields. Thus, it is necessarily a free field theory at low energies, but it could involve higher derivative corrections. In addition to these massless fields, we expect to have for any confining string additional (bosonic and fermionic) degrees of freedom on the worldsheet at some scale m; in a gauge theory characterized by a single scale Λ (like pure Yang-Mills theories) we expect $m \sim \Lambda$ to be of the same order as the square root of the string tension, while in gauge theories with dimensionless parameters there may be some separation between the scales. The theory on the worldsheet at the scale m may be weakly coupled, in which case we can describe the additional degrees of freedom as massive particles, or it could be strongly coupled, in which case we have no such description. The latter is more likely in a theory with a single scale, in which the width of any particle-like state is governed by the same scale as its mass. In either case we expect the effective action to be valid only below the scale m, where it should break down.

For a generic string-like soliton there is no reason to believe that any effective action makes sense above the typical dynamical scale Λ of the field theory. However, the situation of the confining string in large N gauge theories is different, since we believe [1] that such gauge theories are equivalent to weakly coupled string theories, and in such theories there is a well-defined action on the worldsheet that is valid at all energy scales. (Indeed, the quantization of this action should include the full information about the large N gauge

theory.) In such a situation we can think of the low-energy effective action of the massless fields as coming from the exact string worldsheet action, when we integrate out all the massive degrees of freedom on the worldsheet. Note that the action of a fundamental string has a diffeomorphism symmetry, and, depending on the formalism, it may also contain a worldsheet metric as a dynamical variable. Often in string theory we use the conformal gauge, in which the worldsheet action is conformally invariant and there is no mass scale. It is important to emphasize that the effective action on a long string arises in a different gauge, in which we gauge-fix the diffeomorphism symmetry such that two of the worldsheet coordinates are identified with space-time coordinates (the "static gauge"). In this gauge the action has a mass scale, and we typically get (as we will see in various examples) a theory of massive and massless fields. The low-energy effective action discussed above arises when integrating out these massive fields. For strings in flat space, the effective action in this gauge is precisely the Nambu-Goto action, which has only massless fields but includes an infinite tower of higher derivative corrections to their action. This is a special case where the effective action should make sense at all energies; of course, we know that such a string theory is only consistent for a superstring in $D = 10$. Confining strings arise from superstrings in curved backgrounds, and then some of the fields on the worldsheet are massive at some scale m, and the low-energy effective action is more complicated.

As described above, we expect the effective action on a confining string to depend on the derivatives of $(D-2)$ scalar fields, which we will denote by X^i $(i = 2, \cdots, D-1)$. Apriori the action $S = \int d^2\sigma \mathcal{L}(\sigma)$ should include the most general terms consistent with the $SO(D-2)$ rotation symmetry. At 0-derivative order there is a term in the action density proportional to the effective string tension T,

$$\mathcal{L}_0 = -T. \tag{2.1}$$

At 2-derivative order there is a single possible term

$$\mathcal{L}_2 = -\frac{1}{2}\partial^\alpha X \cdot \partial_\alpha X, \tag{2.2}$$

whose coefficient we can always normalize in this way. Here $\alpha = 0, 1$ goes over the worldsheet coordinates, and we use the notation $X \cdot X \equiv X^i X^j \delta_{ij}$. At 4-derivative order there are generally two independent terms (ignoring terms proportional to $\partial^2 X^i$ which can be eliminated by field redefinitions),

$$\mathcal{L}_4 = c_2(\partial^\alpha X \cdot \partial_\alpha X)(\partial^\beta X \cdot \partial_\beta X) + c_3(\partial^\alpha X \cdot \partial^\beta X)(\partial_\alpha X \cdot \partial_\beta X). \tag{2.3}$$

The notation that we use here follows [7], up to a different normalization of c_2 and c_3. In the special case of $D = 3$, there is only one field X and the two terms in (2.3) are identical. At six-derivative order, there are several terms that apparently cannot be eliminated by field redefinitions :

$$\begin{aligned}
\mathcal{L}_6 &= \mathcal{L}_{6,4} + \mathcal{L}_{6,6}, \\
\mathcal{L}_{6,4} &= c_4(\partial_\alpha\partial_\beta X \cdot \partial^\alpha\partial^\beta X)(\partial_\gamma X \cdot \partial^\gamma X) + c_5(\partial_\alpha X \cdot \partial_\beta X)(\partial_\gamma X \cdot \partial_\alpha\partial_\beta\partial_\gamma X), \\
\mathcal{L}_{6,6} &= c_6(\partial_\alpha X \cdot \partial^\alpha X)^3 + c_7(\partial_\alpha X \cdot \partial^\alpha X)(\partial_\beta X \cdot \partial_\gamma X)(\partial^\beta X \cdot \partial^\gamma X) + \\
&\quad c_8(\partial_\alpha X \cdot \partial_\beta X)(\partial^\alpha X \cdot \partial_\gamma X)(\partial^\beta X \cdot \partial^\gamma X).
\end{aligned} \tag{2.4}$$

The c_5 term is naively non-trivial, but in fact since our action lives in a two-dimensional space, one can show that it is actually proportional to the equation of motion (up to integrations by parts); this is most easily seen by using light-cone coordinates, where the leading order equation of motion is $\partial_+\partial_- X^i = 0$. Thus, we will ignore this term from here on. Similarly, in two dimensions the c_8 term can be shown to be equal to a linear combination of the c_6 and c_7 terms[1], so we will ignore it as well. For the special case of $D = 3$ the c_4 term is equivalent to the c_5 term so it is also trivial, and there is only one independent term in $\mathcal{L}_{6,6}$.

The effective action we wrote here is for a string with no boundaries, and then only terms with an even number of derivatives are allowed. In many cases it is interesting to consider also confining strings with boundaries; for instance, such a situation arises in the computation of Wilson loops (which are boundaries for a confining string worldsheet), including the computation of the force between external quarks and anti-quarks. In the presence of boundaries, there could be additional terms in the effective action which are localized on the boundary (and may involve an even or an odd number of derivatives); in particular, in the analysis above we did not write down terms which differ by an integration by parts, so if we make a different choice for the form of the terms we write we will generate some boundary terms. However, it is important to emphasize that the same confining string could have different types of boundaries; for instance the string could end either on a Wilson loop or on a domain wall, and it is not obvious that the boundary terms should be the same for different boundaries. In this paper we focus on the closed string effective action, and on the corrections to the closed string spectrum, so we ignore the boundary terms (which only affect the open string spectrum). In some of our computations we will use worldsheets with boundaries, and we will then assume that there are no boundary terms; this seems reasonable for a string ending on a domain wall described by a D-brane, though it is not necessarily true for strings ending on Wilson loops. This assumption does not influence our results concerning the closed string effective action.

What can we compute using the effective action ? Obviously, we can use the tree-level effective action to compute any dynamical processes below the scale m where the action is expected to break down. However, it would be nice to be able to use the effective action also for loop computations, such as the computation of the partition function of a long string whose worldsheet is (say) a torus (this includes the corrections to the energies of winding closed string states), or loop corrections to scattering amplitudes on the worldsheet. Generic loop computations lead to divergences, so the answer depends on the physics at the cutoff scale and additional information is required (beyond the effective action) to obtain finite answers. However, in special cases loop computations may give finite results, in which case we can trust them. We will see that this happens in many cases when we use the effective action on a superstring; presumably this is because in a different gauge (the conformal gauge) this action is finite, so it should lead to well-defined results. In other cases, like the computation of the partition function of the low-energy effective action on a torus, we will encounter divergences. Of course, such divergences arise already in the

[1]We thank F. Gliozzi for pointing this out to us.

partition function of a free field theory. We will regulate these divergences, as in [9], using a zeta function regularization. This regularization satisfies some nice physical properties (described in [9]) which make it effectively independent of the physics at the cutoff, so we expect it to give correct answers for the physics below the scale m (which should be independent of the cutoff).

2.2 Constraints on the effective action of a confining string

Two types of constraints on the effective action have been considered in the literature. One constraint, originally analyzed by Lüscher and Weisz [7], arises from the fact that the partition function of the string wrapped on various surfaces must have an interpretation in terms of the propagation of physical string states along these surfaces. This constraint is relevant for strings which have a limit in which they are weakly coupled (such as confining strings in $SU(N)$ gauge theories), since in that case the single-string states do not mix (in the weak coupling limit) with any other states, so the partition function has an interpretation involving purely the propagation of single-string states.

A specific case of this, which was considered in [7], involves worldsheets with the topology of an annulus. Suppose that we consider a confining string in a Euclidean space, in which one of the directions is compactified on a circle of circumference L (say, $X^0 \equiv X^0+L$). We can now consider a string whose worldsheet is wrapped once around this circle, and which has two boundaries separated by a distance R in another spatial direction (say, boundaries at $X^1 = 0$ and at $X^1 = R$). For a confining string one example of this is the correlation function of two Wilson loops, wrapped on the circle and separated by a distance R. The partition function $Z^{annulus}(L, R)$ on such a worldsheet has two interpretations. On one hand, we can view X^0 as the "time" direction, and then the diagram is a one-loop vacuum diagram for an open string of length R, which can be expressed as

$$Z^{annulus}(L, R) = \sum_k e^{-E_k^o(R)L}, \tag{2.5}$$

where the sum is over all open string states k of length R, with vanishing transverse momentum (since the ends of the open string are fixed), which have energies $E_k^o(R)$. Note that these energies depend only on R and not on L, since we interpret Z as a statistical mechanical partition function; when we have fermionic states for the string this requires that we put anti-periodic boundary conditions for the fermions in the X^0 direction, otherwise we have an extra factor of $(-1)^F$. For confining strings in pure Yang-Mills theories we do not expect to have any fermionic states so this is not relevant.

On the other hand, we can view X^1 as the "time" direction. Then, we have a closed string state (winding on a circle of circumference L), which is created at $X^1 = 0$ from some "boundary state" and is annihilated at $X^1 = R$. The closed strings can have any transverse momentum, since they propagate between two points with vanishing transverse separation. In other words, if we allow for some transverse separation between the two boundaries and integrate over it, we would sum over only zero transverse momentum closed strings, so we have

$$\int dx_\perp Z^{annulus}(L, \sqrt{R^2 + x_\perp^2}) = \sum_n |v_n(L)|^2 e^{-E_n^c(L)R}, \tag{2.6}$$

where the sum is over all closed string states n with zero transverse momentum, $v_n(L)$ are the overlaps of these states with the boundary state, and $E_n^c(L)$ are their energies. It was shown in [7] that this implies that

$$Z^{annulus}(L,R) = \sum_n |v_n(L)|^2 2R \left(\frac{E_n^c(L)}{2\pi R}\right)^{(D-1)/2} K_{(D-3)/2}(E_n^c(L)R), \qquad (2.7)$$

where $K_\nu(x)$ is a Bessel function. The same equation may be derived from a string theoretic computation of the partition function for a string wrapping the annulus along the lines of [10] (see also [11]), allowing arbitrary energies for the states in the closed string channel.

For very large L and R, the higher derivative corrections to the effective action are negligible, and the energy levels will be those of a free string in D dimensions; this implies that the closed string energy levels are all of the form

$$E_n^{c,L} = E_n^{c,0}(L) + E_n^{c,1}(L) + \cdots = TL + \frac{4\pi}{L}\left[-\frac{1}{24}(D-2) + N_n\right] + \cdots \qquad (2.8)$$

if the state n arises at excitation level $N_n \in \mathbf{Z}$ (this is actually the excitation level for both the right-moving and the left-moving states on the worldsheet; the two are equal for any state that has an overlap with the boundary state). The first term is the classical string energy, and the second is the well-known Lüscher term [6]. We expect the higher derivative corrections to the action to give corrections to (2.8) which, on dimensional grounds, begin at order $1/L^3$; in particular, in a flat-space string theory, which is well-described by the Nambu-Goto action, the exact formula for the energy levels of zero-momentum states is given by

$$E_n^{c,NG}(L) = \sqrt{(TL)^2 + 8\pi T\left[-\frac{1}{24}(D-2) + N_n\right]}, \qquad (2.9)$$

but we do not expect this equation to be exact for general confining strings[2]. Similarly, for large R the open string energy levels take the form

$$E_k^o(R) = E_k^{o,0}(R) + E_k^{o,1}(R) + \cdots = TR + \frac{\pi}{R}\left[-\frac{1}{24}(D-2) + N_k\right] + \cdots, \qquad (2.10)$$

for levels $N_k \in \mathbf{Z}$, with corrections starting at order $1/R^3$. Note that the form of the effective action guarantees that closed string energy levels have an expansion involving purely odd powers of $1/L$. For open strings the same is true if there are no boundary terms, but for strings stretched between a quark and an anti-quark there could be boundary terms which introduce also even powers of $1/R$.

The partition function on the annulus that we compute must be consistent with the two forms (2.5) and (2.7) above. The partition function for large L, R in both channels may be expanded in a power series in inverse powers of L and R (multiplying the exponential terms), which is really an expansion in $E^1, E^2, \cdots$ (where we take $E^k(R)$ to scale

[2] In particular, we expect the ground state energy of the winding string to go to zero at $L = 1/T_H$ where T_H is the Hagedorn temperature, and we expect this temperature in general to be smaller than the one of the Nambu-Goto string, $T_H^{NG} = \sqrt{3T/(D-2)\pi}$. This is confirmed by lattice simulations [12].

as $1/R^{2k-1}$). The comparison to the effective field theory partition function turns out to be simplest if we expand around an expression in which we write both E^0 and E^1 in the exponent, but expand just around E^0 in other places, since this is what we will find in the effective field theory partition function in the free field approximation. In the "open string channel" we then have at the leading non-trivial order

$$Z^{annulus}(L,R) = \sum_k e^{-(E_k^{o,0}(R)+E_k^{o,1}(R))L}(1 - E_k^{o,2}(R)L + \cdots). \tag{2.11}$$

So, in this channel, the leading order correction to the partition function should look like L times a sum over corrections to energies (scaling as $1/R^3$) times exponentials. In the "closed string channel" we obtain to leading non-trivial order (up to two inverse powers of lengths)

$$\begin{aligned} Z^{annulus} \quad & (L,R) = \sum_n |v_n(L)|^2 \left(\frac{E_n^{c,0}(L)}{2\pi R}\right)^{(D-2)/2} e^{-(E_n^{c,0}(L)+E_n^{c,1}(L))R} \cdot \\ & \left(1 - E_n^{c,2}(L)R + \frac{D-2}{2}\frac{E_n^{c,1}(L)}{E_n^{c,0}(L)} + \cdots\right) \cdot \left[1 + \frac{(D-2)(D-4)}{8E_n^{c,0}(L)R} + \cdots\right] \end{aligned} \tag{2.12}$$

We will analyze this expansion in detail in the next section. The leading correction here is more complicated, involving terms scaling as R/L^3, $1/L^2$ and $1/LR$. A $1/L^2$ contribution can also arise by expanding $v_n(L)$ in inverse powers of L.

The method of [7] to constrain the partition function is to first compute the partition function coming from the effective action described in the previous subsection, and then to try to match it to the equations in the previous paragraph (for some corrections to the energy levels and to $v_n(L)$). Of course, the partition function should only match for states whose excitation energies are much smaller than the scale m where the effective action breaks down, but this is true for any state for large enough L and R. In [7] this matching was performed for the 4-derivative terms, by expanding the partition function to leading order in c_2, c_3, and writing it using exponentials either of L/R or of R/L (modular transformation properties of the resulting partition function can be used to relate the two). We will review this computation in section 3. In the "open string channel" this gave reasonable results for any c_2, c_3 (with some specific corrections to open string energy levels at order $1/R^3$, linear in c_2 and c_3). However, in the "closed string channel", the corrections of order $1/LR$, coming from the last parentheses in (2.12), matched only when

$$(D-2)c_2 + c_3 = \frac{D-4}{8T}. \tag{2.13}$$

Thus, they concluded that the effective action is only consistent when this constraint is satisfied. When it is satisfied, the correction takes the form (2.12) and one can compute the leading corrections to the energy levels (and to $v_n(L)$) in both the open and closed string channels. Note that the Nambu-Goto action (whose quantum open string spectrum was computed in [13]), with $c_2^{NG} = 1/8T$ and $c_3^{NG} = -1/4T$, satisfies this constraint, as it has to since one can check that it leads to a partition function on the annulus which is

consistent in both channels (to all orders). For $D = 3$, since there is only a single four-derivative coefficient, the constraint (2.13) implies that the four-derivative action must agree with the Nambu-Goto action. For $D > 3$ there is one free coefficient remaining, so it seems that the action (and the energy levels) need not agree with Nambu-Goto at the four-derivative order (namely, for corrections to energies of order $1/R^3$ or $1/L^3$). In section 3 we will extend the considerations of this paragraph to the next order in the derivative expansion.

In [7] they also showed that no boundary terms can contribute up to two-derivative order; here, as mentioned above, we will not discuss the boundary terms, but we will just take them to zero (assuming that there is some consistent boundary of the confining string which gives this). It would be interesting to generalize our analysis to obtain constraints on boundary terms involving higher derivatives than considered in [7].

A similar analysis may be performed by considering a string wrapping a torus in space-time, which we will take for simplicity to involve two orthogonal periodic coordinates, $X^0 \equiv X^0 + L$ and $X^1 \equiv X^1 + R$. Obviously, the partition function in this case must be invariant under $L \leftrightarrow R$ (modular invariant), and this is automatically satisfied for any effective action. However, it must also have an interpretation as the sum over closed string states winding the X^1 circle, propagating for a time L, or the other way around. By similar arguments to the ones above (or by a simple generalization of the computation of the torus partition function in [10], see also [14]), this partition function must take the form (up to an unimportant constant depending on the radii and on the transverse volume)

$$Z^{torus}(L,R) = \sum_n R \left(\frac{E_n^c(L)}{R}\right)^{(D-1)/2} K_{(D-1)/2}(E_n^c(L)R), \tag{2.14}$$

where the states summed over here are all the closed string states winding once around the circle with zero transverse momentum (including states with different numbers of left and right-moving excitations on the worldsheet, which carry a non-zero momentum around the worldsheet). Again, we can expand this in inverse powers of L and R, and compare to the expressions that we obtain from the low-energy effective action. Here we have less freedom (since there are no unknown coefficients $v_n(L)$), so we will find more constraints. We will see in section 3 that at the leading non-trivial order this will allow us to uniquely determine c_2 and c_3 for any D, such that they must equal the Nambu-Goto values, and we will extend the analysis to the next order as well.

Before we conclude this section, it is important to stress the assumptions that go into the constraints discussed above on the string effective action. One assumption is that the string has a limit in which it is weakly coupled. If the string is not weakly coupled, there is no physical observable that gets contributions only from single-string states as we assumed above, since there is generically mixing between single-string and multi-string states. It is not clear how this mixing affects the energy levels of winding states; it would be interesting to analyze this. Thus, for confining $SU(N)$ strings, we expect the effective action on the worldsheet to obey the constraints above (since there is a large N limit where the string theory is weakly coupled), but for finite N the energy levels obtain corrections from mixing so there may be deviations from the large N predictions derived above. A second

assumption, which goes into the way we regularize the loop diagrams in the worldsheet effective action, is that there is no dependence of the results on any UV cutoff scale. This assumption is presumably true for solitonic strings that correspond to standard weakly coupled string theories, since in such theories there is a gauge where the worldsheet theory is conformal, and there is no dependence on any high energy scale. However, generic solitonic strings may not correspond to any weakly coupled string theory, and for such strings it is not clear that physics at high energies decouples from the low-energy physics captured by the effective action. Thus, our predictions for the form of the effective action apply to confining strings (in the fundamental representation) in large N gauge theories, but a priori it is not clear whether they should hold more generally or not.

Even with these assumptions it seems that we are getting constraints for "free", but we are really getting them by using the symmetries of the problem. Specifically, our theories have a $SO(D-1,1)$ Lorentz symmetry, but the effective action on the string is only explicitly invariant under an $SO(D-2)$ subgroup. The derivations of the expressions (2.7) and (2.14) use the rotation symmetry between the transverse coordinates and the coordinate which the string propagates in, so our constraints really test this $SO(D-2,1)$ symmetry (this was explicitly shown in [8] for the annulus). It would be interesting to derive the constraints above directly from Ward identities for the broken symmetries, and to check whether additional constraints may be derived by using the full Poincaré symmetry. Note that the Nambu-Goto string, whose full spectrum is not Poincaré-invariant (for $D \neq 26$, since the massive states do not all lie in representations of $SO(D-1)$), still satisfies all the constraints that we discuss here.

A different type of constraint on the effective action was considered by Polchinski and Strominger in [2]. They used the fact that the confining string is expected to be (at least in the large N limit [1]) a fundamental string in some curved background. This implies that one can go to a "conformal gauge" in which the theory on the worldsheet of the string must be a conformal theory with a specific central charge ($c = 26$ if it is a bosonic string, or $c = 15$ if it is a superstring). They attempted to write down such a conformal theory for the D scalar fields describing the position of the string in $\mathbf{R}^D$ (since they use a conformal gauge they cannot go to static gauge, so the action involves all the coordinates). The action they wrote down is singular (it involves negative powers of $(\partial X \cdot \partial X)$), but becomes non-singular when expanded around a long string configuration of the type described above; we can interpret this by saying that their action may be derived by integrating out all the other degrees of freedom on the string, and this integrating out is useful (gives a non-singular action) when expanding around a long string configuration in which these additional degrees of freedom are heavy. In this formalism an expansion similar to, but not identical to, the derivative expansion described above was obtained in [2]. They showed that the leading correction to the free action in this expansion is uniquely determined, and it was later shown in [3, 4] that, for any D, this implies that the leading correction to the string ground state energy (at order $1/L^3$) is the same as in the Nambu-Goto action. Of course, this, as well as the claim that the leading correction to the effective action is unique (and equal to Nambu-Goto), is consistent with the constraints we described above.

It is not clear to us precisely how to relate the effective action in the Polchinski-

Strominger formalism to the one in the Lüscher-Weisz formalism – it would be very interesting to understand this. In particular the Polchinski-Strominger formalism seems to involve the full Poincaré symmetry (since it is only valid for a string with the critical central charge) which is not used in the Lüscher-Weisz formalism. It is not completely obvious to us if the Polchinski-Strominger formalism is valid or not (namely, if integrating out the other fields on the string indeed always gives an effective action of the form that they assume). If it is valid, it would be interesting to extend it to the next order, in order to see if this leads to more or to less constraints on the corrections to the effective action than the ones that we derive from the Lüscher-Weisz formalism. This is not clear, since the assumptions that underlie the two formalisms do not seem to be identical.

2.3 The effective string action in weakly curved holographic backgrounds

As already mentioned above, we believe that the confining string (in the fundamental representation) in a large N gauge theory is equivalent to a weakly coupled fundamental string in some background, by a generalization of the AdS/CFT correspondence [15, 16, 17]. In principle, given such a background, we can write down the string action for a long string configuration in static gauge[3]. As mentioned above, we expect that all degrees of freedom except the $(D-2)$ Nambu-Goldstone bosons will have some mass m in this background, and, thus, we can integrate them out to obtain the effective action below the scale m. Recall that already for a string in flat space, the Nambu-Goto action is an effective action expanded in derivatives divided by the string scale $M_s \sim \sqrt{T}$ so we expect it to break down (or become strongly coupled) at this scale. For a generic gauge theory, with no dimensionless parameters, we expect $m \sim M_s$. This means that the effective action at the scale m will generically be strongly coupled, and it is not clear how to integrate out the degrees of freedom at this scale.

However, for a special class of gauge theories, the dual string background is weakly curved (such a background for a superstring is necessarily ten dimensional). Several examples of this class were discovered in the last decade [18, 19, 20, 21]. The fact that the background is weakly curved means that the string tension is much larger than the typical curvature scale of the background; the latter determines the masses of the additional fields on the worldsheet, so that in such a case we have $m^2 \ll T$. The effective theory on the worldsheet at the scale m is then weakly coupled – the dimensionless coupling constant is m^2/T – and we can perturbatively integrate out the massive fields to obtain the low-energy effective action of the massless fields. We will do this in detail in section 6. In these backgrounds we naively expect the effective action to be equal to the Nambu-Goto action, with corrections that are a power series in m^2/T (coming from loops of massive string states). Note that usually when massive states are integrated out, the corrections to the effective action go as a negative power of their mass, while here the corrections go as a positive power of the mass. This is because the massless limit corresponds exactly to a string in flat space (the Nambu-Goto action), so the deviations from this limit must go as

[3] In the last 12 years it has been realized that also non-confining gauge theories have a string theory description. However, in these theories there is no classical long closed string configuration of the type we analyze in this paper, and no expansion in the inverse string tension.

a positive power of the mass; in practice we will see that this arises because the couplings of the additional fields to the massless fields will contain powers of m.

As mentioned above, in typical gauge theories (like pure Yang-Mills theory) we do not expect m to be small, so we cannot perform such a power series expansion of the corrections. Our goal will be to see which terms in the action deviate from the Nambu-Goto form in the regime of small m and which do not; it seems plausible that any terms that are allowed to deviate, will do so already at leading order, and thus we conjecture that the same terms should deviate from Nambu-Goto also in general gauge theories (though in general we expect any allowed deviations to be of order one in string units). We will find that indeed our one-loop effective action will take the most general form allowed by the constraints discussed in the previous subsection, up to one additional constraint that it satisfies.

A special example of such a computation of corrections to the action, which was already analyzed in the literature, is the computation of the effective tension of the string; the classical tension receives corrections at one-loop from integrating out the massive fields, and it was found in [22] that the corrected tension takes the form

$$T' = T + \frac{1}{8\pi}\left(\sum_{fermions\ F} m_F^2 \log(m_F^2) - \sum_{bosons\ B} m_B^2 \log(m_B^2)\right), \tag{2.15}$$

where T is the classical tension, and m_B and m_F are the masses of the bosonic and fermionic degrees of freedom on the worldsheet, respectively. Note that we generally expect to have $(10-D)$ massive scalars on the worldsheet (since we are dealing with a string in a background which is ten dimensional), and 8 massive fermions (since this is the number of physical fermions living on a superstring), though in some cases some of these fields may be massless (and do not contribute to (2.15)). Note also that the logarithms appearing in this formula are really of the form $\log(m^2/\Lambda^2)$ where Λ is the UV cutoff, so in order for (2.15) to be finite, it must be the case that

$$\sum_{bosons\ B} m_B^2 = \sum_{fermions\ F} m_F^2. \tag{2.16}$$

This is true in all known holographic backgrounds, and it seems to be necessary to obtain finite (cutoff-independent) results for various worldsheet computations (as we expect, since the theory in conformal gauge is manifestly independent of the cutoff). We will assume from here on that equation (2.16) holds.

In all known examples of confining string backgrounds, we actually find more massless scalars than expected on the worldsheet. This is because in these backgrounds the confining string lives in some "IR" region of space, and in this region there is some p-dimensional sphere that the string is localized on (and there is an exact or approximate $SO(p+1)$ symmetry), which generally carries some non-zero flux which stabilizes the background. The string is localized on this sphere, meaning that it has p additional massless fields e^j on its worldvolume corresponding to its position on the sphere. Generally, upon integrating out the massive fields (including the fermions), the effective action of these fields looks like

a sigma model on S^p. So, we expect that perturbatively the e^j fields will remain massless, but that non-perturbatively this sigma model will develop a mass gap, at some scale $\tilde{\Lambda}$ which is exponentially smaller than the scale m (the radius of the sphere is of order $1/m$, so we expect $\tilde{\Lambda} \sim m \exp(-CT/m^2)$ for some constant C). In such a case, in the effective action for length scales between $1/\tilde{\Lambda}$ and $1/m$ we should include the e fields as well, and only in the effective action at length scales above $1/\tilde{\Lambda}$ we can integrate them out. However, the field theory of the e's is strongly coupled, so this will lead to various corrections to the action depending on the scale $\tilde{\Lambda}$ that we do not know how to compute.

Our attitude will be to ignore the e fields in our computations, and only to integrate out the fields which obtain a mass at the curvature scale m. For the effective action between the length scales $1/\tilde{\Lambda}$ and $1/m$, this means that we will reliably compute the terms in the effective action that depend only on the X^i, but that the full effective action will also contain the e fields and various couplings involving them (which we will not compute, though they can easily be computed). In the effective action at very low energies (below $\tilde{\Lambda}$), there will be additional terms coming from integrating out also the e's, which we do not know how to compute. However, these terms will depend on $\tilde{\Lambda}$ rather than on the scale m, so they will be parameterically separated from the terms that we do compute. Namely, we compute all the terms in the effective action that depend on the scale m and not on other scales. Our main interest will be in seeing for which terms in the effective action deviations from Nambu-Goto exist; if we find a deviation from Nambu-Goto associated with the scale m then we can be sure that this deviation remains non-vanishing also after adding the contributions from the e fields, since their contributions involve different scales.

Since we are only interested in effects depending on m, we can also ignore other contributions to the effective action which are independent of m, such as effects coming from ghost loops and loops of the worldsheet metric (at least at the one-loop order that we will be working in). Since we ignore metric loops, it will not matter whether we work in the Polyakov or in the Nambu-Goto formalism. However, the difference between the two formalisms may be important at higher orders.

Note that in the holographic description, we can introduce a boundary to the string worldsheet in two ways. One possibility is to compute a Wilson loop (or a correlation function of Wilson loops); in this case the fundamental string worldsheet ends on the boundary of the higher dimensional space (it goes to infinity in the radial direction) at the position of the Wilson loop [23, 24]. The worldsheet in this case does not sit just in the "IR" region of space, so the action in this case may well have boundary terms which depend on what happens throughout the holographic dual space-time. The second possibility is that the string can end on a D-brane; D-branes can describe various objects in a confining string theory, including dynamical quarks and domain walls. In this case, if the D-brane extends into the "IR" region of space so that the open string can be completely localized in this region, we do not expect to get any boundary terms on the worldsheet, since there are no such terms for a string ending on a D-brane.

2.4 A special class of holographic backgrounds

In general weakly curved holographic backgrounds, we obtain small corrections to the

Nambu-Goto action as described above. In these backgrounds the confining string is described as a fundamental string moving in some curved space with various p-form fields (typically including Ramond-Ramond (R-R) fields) turned on. We expect some such background (weakly curved or not) to correspond to any large N gauge theory in the 't Hooft limit. However, there is also a class of large N confining gauge theories whose confining string is not well-described as a weakly coupled fundamental string as above; these do not arise from a 't Hooft large N limit but from a different type of limit. As an example of this, consider the background of N D5-branes compactified on a two-sphere, in the limit where the theory on the D5-branes is decoupled from the string theory in the bulk; the gravity solution describing this background was found by Chamseddine and Volkov in [25] and interpreted in this way by Maldacena and Nuñez in [19]. In the decoupling limit the $5+1$ dimensional effective Yang-Mills coupling g_6 is kept fixed, and there is a dimensionless parameter corresponding to the size of the two-sphere in units of the 't Hooft coupling $g_6^2 N$. (Of course, the six dimensional Yang-Mills theory requires some UV completion, and this is provided in this decoupling limit by a "little string theory".) In the limit where this size is small, the background is weakly coupled in an S-dual frame to the original frame, such that the confining string is best described by a D-string moving in a weakly curved background [4]. At first sight this case seems very similar to the case of a fundamental string; the effective action on the D-string, which is the DBI action, is essentially the same as the Nambu-Goto action, and again there are some fields on the long D-string worldsheet that are massive and that may be integrated out. However, the fact that the D-string action has an inverse string coupling $1/g_s$ multiplying it, means that corrections from loops of the massive states are down by powers of g_s compared to the classical action. Additional corrections of the same order come from string diagrams of higher genus, which have not been computed (recall that the DBI action just captures the string disk diagrams). So, in this case the effective action on the confining string worldsheet, at leading order in g_s, is precisely the same as the Nambu-Goto action, with no deviations at all. At the next order in g_s there will be deviations, but computing them requires higher genus diagrams so it is rather complicated. This case is thus rather different from the standard confining string case discussed in the previous subsection, where we expect any allowed deviations from Nambu-Goto to appear already at leading order in $g_s \sim 1/N$.

2.5 A brief review of lattice results

The effective action of confining strings on the lattice has been studied in the last few years with increasing accuracy (see [26] for a review of results before the ones we explicitly discuss below). In particular, for the three dimensional pure Yang-Mills theory, very precise results have been obtained for the spectrum of closed confining strings in large N $SU(N)$ gauge theories (in the fundamental representation) winding on a circle of circumference L [27, 28]. For the ground state energy, it has been found that it agrees with the Nambu-Goto

[4] There is also a range of values of the size for which the IR region of the background is weakly coupled in the original frame, and there the confining string is described by a fundamental string, but the coupling becomes strong away from the IR region. For this range of values the confining string behaves in the generic way that we expect for large N gauge theories, see section 5.3.

result at order $1/L$ (the Lüscher term) and is consistent with it at order $1/L^3$, and that if there is a deviation at order $1/L^5$ its coefficient is very small (so that the deviation may well be at a higher order in $1/L$). Excited states seem to again agree with Nambu-Goto at orders $1/L$ and $1/L^3$, but to deviate more from it at higher orders, perhaps already at order $1/L^5$ (the deviations seem larger than those of the ground state, but the lattice data is not precise enough to determine at which order it occurs). In $3+1$ dimensions the simulations of large N gauge theories again find agreement with Nambu-Goto for large L, but they are not yet precise enough to tell at what order deviations from Nambu-Goto arise. Simulations of interfaces in the $2+1$-dimensional Ising model similarly show good agreement with Nambu-Goto (see, for instance, [29] and references therein), and again they are not yet precise enough to tell at what order deviations from Nambu-Goto arise.

Recently, $2+1$ dimensional confining strings in higher representations ("k-strings") were also studied [30, 31] in the large N limit and compared to Nambu-Goto, and it was found that they exhibit larger deviations from Nambu-Goto (which may already start at order $1/L^5$) for all states, including the ground state. However, it is subtle to interpret these results, for the following reason. In the large N limit (with fixed k), the binding energy of k fundamental strings to form a k-string goes to zero as $1/N^\alpha$; some theoretical arguments suggest that $\alpha = 2$, while other arguments (see, for instance, [32]) and lattice results suggest that $\alpha = 1$. This means that in the large N limit there are (at least) $(k-1)(D-2)$ light modes on the worldvolume of a k-string, whose mass goes to zero in the large N limit as $1/N^{\alpha/2}$. The general constraints above concerning the form of the effective action are valid only at length scales larger than the inverse of any mass of a worldsheet field (and also large enough so that there is no mixing of the k-string states with states of k fundamental strings); thus, they only apply for length scales bigger than $N^{\alpha/2}/\sqrt{T}$, and it is not clear if these scales are accessed by the simulations yet (with enough precision to extract the $1/L^5$ corrections to energy levels). It would be interesting to analyze k-strings at longer scales, to see if they obey our constraints at these scales.

In another recent paper [33], the energy of the ground state of a confining string in a $2+1$ dimensional system arising from a continuum limit of percolation was numerically computed, and found to agree again with Nambu-Goto at orders $1/L$ and $1/L^3$, but to significantly deviate from it at order $1/L^5$ (though it is not clear if there are enough data points within the range where the $1/L$ expansion can be trusted in order to be able to reliably compute this). Since this model does not have any obvious large N limit where it corresponds to a weakly coupled string theory, it is not clear if the general arguments above apply to it or not; as we will see below, the results of [33] seem to contradict the general predictions for an effective action of a string which has a weakly coupled limit.

3. The effective theory on a confining string

In this section we compute the partition function for the general effective action (2.1)-(2.4) at six derivative order, both on the annulus and on the torus. The computation is perturbative, and the partition function is a power series in $(\sqrt{T}L)^{-1}$ and $(\sqrt{T}R)^{-1}$. We obtain expressions such as the Dedekind function and its derivatives. This allows us to

expand the results both in exponents of L/R (times powers) and in exponents of R/L (times powers). On the annulus this corresponds to the closed and open string channels, while on the torus both limits correspond to closed string partition functions. We require the partition function to have the form (2.7) for the annulus and (2.14) for the torus, when expanded in the long string limit. From this we can extract the open and closed string spectrum. Most importantly, we discover that the coefficients $c_2, \cdots, c_7$ are not arbitrary.

In each subsection we start by writing the general form of the partition function, given in section 2, which is relevant at that order. We then compute, perturbatively, the partition function, using the action $S = S_0 + S_2 + S_4 + S_6$, and compare the two.

3.1 Partition function at $O(T^0)$

To leading order, the expressions (2.5), (2.7) and (2.14) take the following forms :

$$
\begin{aligned}
Z^{ann.} &= \sum_{k=0}^{\infty} \omega_k e^{-(E_k^{o,0}+E_k^{o,1})L} = \sum_{n=0,2,4,\cdots}^{\infty} |v_n^0|^2 \left(\frac{TL}{2\pi R}\right)^{\frac{1}{2}(D-2)} e^{-(E_n^{c,0}+E_n^{c,1})R} \ , \\
Z^{tor.} &= \sum_{n=0}^{\infty} \sqrt{\frac{\pi}{2}} \left(\frac{TL}{R}\right)^{\frac{1}{2}(D-2)} \omega_n^{tor.} e^{-(E_n^{c,0}+E_n^{c,1})R} \ .
\end{aligned} \tag{3.1}
$$

The summations are over energy levels $n = N_L + N_R$, where $N_{L(R)}$ are the number of left(right)-moving excitations on the worldsheet. ω_n is a weight factor corresponding to the number of states at each energy level (we join together the contributions from different states with the same energies, anticipating a degeneracy in the leading order). The annulus boundary carries zero momentum in the compact direction, and so the annulus partition function contains only states with an equal number of left-moving and right-moving excitations, which explains the summation over even n in our equations. We use the closed (open) energy level expansion $E_n^{c(o)} = E_n^{c(o),0} + E_n^{c(o),1} + \cdots$, and $v_n(L) = v_n^0(L) + v_n^1(L) + \cdots$ are the overlaps between the boundary state and the closed string states at level n. More precisely, $|v_n|^2 = \sum_{i\in n} |v_{n,i}|^2$, where $v_{n,i}$ is the overlap with a specific state i at level n.

From the worldsheet effective action point of view, we derive the partition function at this order from the free action $S = S_0 + S_2$. It is convenient to write the partition function using the definitions

$$
\begin{aligned}
q &\equiv e^{2\pi i \tau} \quad , \quad \tilde{q} \equiv e^{2\pi i \tilde{\tau}} \quad , \\
\tau_{ann.} &= i\frac{L}{2R} \quad , \quad \tilde{\tau}_{ann.} \equiv -\frac{1}{\tau_{ann.}} = i\frac{2R}{L} \quad , \quad \tau_{tor.} = i\frac{L}{R} \quad , \quad \tilde{\tau}_{tor.} = i\frac{R}{L} \quad .
\end{aligned} \tag{3.2}
$$

The partition function (up to constants including the transverse volume in the torus case)

is [9]:

$$
\begin{aligned}
Z_0^{ann.} &= e^{-TLR}\eta(q^{ann.})^{2-D} = \sum_{k=0}^{\infty}\omega_k e^{-TLR-\frac{\pi L}{R}[-\frac{1}{24}(D-2)+k]} \\
&= \left(\frac{2R}{L}\right)^{\frac{2-D}{2}} \sum_{n=0,2,4,\cdots}^{\infty} \omega_n^{ann.} e^{-TLR-\frac{4\pi R}{L}[-\frac{1}{24}(D-2)+\frac{n}{2}]} , \\
Z_0^{tor.} &= R^{2-D}e^{-TLR}\eta(\tilde{q}^{tor.})^{2(2-D)} = R^{2-D}\sum_{n=0}^{\infty}\omega_n^{tor.} e^{-TLR-\frac{4\pi R}{L}[-\frac{1}{24}(D-2)+\frac{n}{2}]} .
\end{aligned} \tag{3.3}
$$

Here ω_n are weight factors, proportional to the number of states at each energy level n, and we also used the modular properties of the Dedekind eta function,

$$
\eta(q) \equiv q^{1/24}\Pi_{n=1}^{\infty}(1-q^n), \quad \eta(q) = \sqrt{\frac{i}{\tau}}\eta(\tilde{q}) . \tag{3.4}
$$

By matching (3.3) and (3.1) we find the energies and overlap functions to first order. For the annulus,

$$
\begin{aligned}
E_n^{c,0} &= TL, \quad E_n^{c,1} = \frac{4\pi}{L}[-\frac{1}{24}(D-2)+\frac{n}{2}], \\
|v_n^0|^2 &= \omega_n^{ann.}\left(\frac{\pi}{T}\right)^{\frac{1}{2}(D-2)} ,
\end{aligned} \tag{3.5}
$$

and the open string energies are given in (2.10). The same closed string energies appear in the torus partition function, so that the annulus and torus computations are consistent.

3.2 Partition function at $O(T^{-1})$

We carry on with the expansion of the partition function to order $O(T^{-1})$. We explicitly use our results from the previous subsection. By expanding equations (2.7) and (2.14) we expect to obtain expressions of the form

$$
\begin{aligned}
Z^{ann.} &= \sum_{n=0,2,4,\cdots}^{\infty} |v_n^0|^2 \left(\frac{TL}{2\pi R}\right)^{\frac{1}{2}(D-2)} e^{-(E_n^{c,0}+E_n^{c,1})R}\{1-[E_n^{c,2}]_{ann.}R + \frac{|v_n^1|^2}{|v_n^0|^2} \\
&\quad + \frac{2\pi(D-2)}{TL^2}[-\frac{1}{24}(D-2)+\frac{n}{2}] + \frac{(D-2)(D-4)}{8TLR} + \cdots\} , \\
Z^{tor.} &= \sum_{n=0}^{\infty}\sqrt{\frac{\pi}{2}}\left(\frac{TL}{R}\right)^{\frac{1}{2}(D-2)} e^{-(E_n^{c,0}+E_n^{c,1})R}\omega_n^{tor.}\{1-[E_n^{c,2}]_{tor.}R + \frac{D(D-2)}{8TLR} \\
&\quad + \frac{2\pi(D-2)}{TL^2}[-\frac{1}{24}(D-2)+\frac{n}{2}] + \cdots\} .
\end{aligned} \tag{3.6}
$$

The notation $[E_n^{c,2}]$ indicates there is an averaging over all states at level n, for the torus with equal weight and for the annulus with weight $|v_{n,i}|^2$.

We now compute the partition function of the action $S = S_0+S_2+S_4$. There are two 2-loop bubble diagrams (see figure 1), with the vertices c_2 and c_3. Two possible contractions

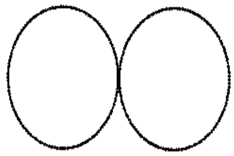

Figure 1: The 2-loop contribution to the partition function at $O(T^{-1})$.

in each diagram lead to expressions proportional to $(D-2)^2$ and $(D-2)$. The computation of the diagrams gives

$$\begin{aligned}\langle S_4\rangle_0 \,&=\, c_2[(D-2)^2\times I_1+2(D-2)\times I_2] \\ &\quad +c_3[(D-2)\times I_1+((D-2)^2+(D-2))\times I_2],\end{aligned} \tag{3.7}$$

with

$$I_1 \,=\, \int d^2\sigma\partial_\alpha\partial^{\alpha'}G\partial_\beta\partial^{\beta'}G, \quad I_2=\int d^2\sigma\partial_\alpha\partial_{\beta'}G\partial^\alpha\partial^{\beta'}G\ . \tag{3.8}$$

Here $G=\lim_{\sigma\to\sigma'}G(\sigma,\sigma')$ is the propagator of the free field X^i in coordinate space. We compute the diagrams in detail in appendix A, as originally done in [7, 9]. The computation is rather straightforward, apart from the need to carefully use a consistent zeta function regularization. We find the following result for the annulus, expressed using the Eisenstein series $E_n(q)$ and their derivatives $H_{n,k}(q)$, all defined in section A.1 :

$$\begin{aligned}I_1^{ann.} \,&=\, \frac{2\pi^2L}{R^3}H_{2,2}(q^{ann.})=-\frac{1}{LR}+\frac{2\pi}{3L^2}E_2(\tilde{q}^{ann.})+\frac{32\pi^2R}{L^3}H_{2,2}(\tilde{q}^{ann.})\ , \\ I_2^{ann.} \,&=\, \frac{\pi^2L}{R^3}[\frac{2}{(24)^2}E_2^2(q^{ann.})+H_{2,2}(q^{ann.})]=\frac{16\pi^2R}{L^3}[\frac{2}{(24)^2}E_2(\tilde{q}^{ann.})^2+H_{2,2}(\tilde{q}^{ann.})]\ .\end{aligned} \tag{3.9}$$

For the torus we find (a similar computation for the Nambu-Goto action was performed in [14])

$$\begin{aligned}I_1^{tor.} \,&=\, \frac{1}{RL}\ , \\ I_2^{tor.} \,&=\, \frac{\pi^2R}{18L^3}E_2^2(\tilde{q}^{tor.})-\frac{\pi}{3L^2}E_2(\tilde{q}^{tor.})+\frac{1}{RL}\ .\end{aligned} \tag{3.10}$$

The partition function at this order is given by,

$$\begin{aligned}Z(q) \,&=\, Z_0(q)(1-\langle S_4\rangle_0) \\ &\propto \sum_{n=0}^{\infty}\omega_n e^{-R(E_n^{c,0}+E_n^{c,1})}\{1-I_1[(D-2)^2c_2+(D-2)c_3] \\ &\qquad -I_2[(D-2)(D-1)c_3+2(D-2)c_2]\}\ .\end{aligned} \tag{3.11}$$

In the annulus case we see that the corrections to the open string partition function are all energy corrections proportional to L/R^3, as expected from (2.5). Plugging our results

(3.9), (3.10) in (3.11) gives the following partition functions for closed strings,

$$
\begin{aligned}
Z^{ann.} &= \left(\frac{L}{2R}\right)^{\frac{1}{2}(D-2)} \sum_{n=0,2,4,\cdots}^{\infty} \omega_n^{ann.} e^{-R(E_n^{c,0}+E_n^{c,1})} \cdot \\
&\quad \times \{1 - \frac{\pi^2 R}{18L^3} E_2(\tilde{q}^{ann.})^2 [(D-2)(D-1)c_3 + 2(D-2)c_2] \\
&\quad + \left(\frac{1}{LR} - \frac{2\pi}{3L^2} E_2(\tilde{q}^{ann.})\right) [(D-2)^2 c_2 + (D-2)c_3] \\
&\quad - \frac{16\pi^2 R}{L^3} H_{2,2}(\tilde{q}^{ann.}) [(D-2)(D+1)c_3 + 2(D-2)(D-1)c_2]\}, \\
Z^{tor.} &= R^{2-D} \sum_{n=0}^{\infty} \omega_n^{tor.} e^{-R(E_n^{c,0}+E_n^{c,1})} \cdot \{1 + \left(\frac{\pi}{3L^2} E_2(\tilde{q}^{tor.}) - \frac{\pi^2 R}{18L^3} E_2^2(\tilde{q}^{tor.})\right) \\
&\quad \times [(D-2)(D-1)c_3 + 2(D-2)c_2] - \frac{1}{RL}(D-2)D(c_3+c_2)\}.
\end{aligned} \tag{3.12}
$$

We can now compare our result to (3.6). By looking at the ground state ($n = 0$) we find the following constraint from the $\frac{1}{LR}$ term in the annulus partition function [7] :

$$
\frac{D-4}{8T} = (D-2)c_2 + c_3 \ , \tag{3.13}
$$

and we also find,

$$
[E_0^{c,2}]_{ann.} = \frac{\pi^2}{18L^3}(D-2)[(D-1)c_3 + 2c_2] \ , \ \frac{|v_0^1|^2}{|v_0^0|^2} = -\frac{2\pi}{3L^2}(D-2)((D-2)c_2 + c_3) \tag{3.14}
$$

Comparing other states does not give us additional constraints, as shown in [7].

The result from the torus is consistent with the results above, and we also get one additional constraint due to the fact that there are no unknown overlap functions, so we can compare both the $\frac{1}{LR}$ and the $\frac{1}{L^2}$ terms:

$$
\begin{aligned}
c_2 + c_3 &= -\frac{1}{8T} \ , \ -\frac{D-2}{4T} = (D-1)c_3 + 2c_2 \ , \\
[E_0^{c,2}]_{tor.} &= \frac{\pi^2}{18L^3}(D-2)[(D-1)c_3 + 2c_2] = -\frac{\pi^2}{72TL^3}(D-2)^2 \ .
\end{aligned} \tag{3.15}
$$

There are two independent constraints in total, which completely fix the effective action at this order to be the Nambu-Goto action for any D:

$$
c_2 = c_2^{NG} = \frac{1}{8T} \ , \ c_3 = c_3^{NG} = -\frac{1}{4T} \ . \tag{3.16}
$$

In particular, this implies that the partition function (or any other physical observable) is constrained to be the one given by the Nambu-Goto action to this order. Again one can check that higher n's do not give additional constraints, but just give formulas for $[E_n^{c,2}]_{tor.}$ for each level n.

3.3 Partition function at $O(T^{-2})$

At order $O(T^{-2})$ there are numerous contributions to the partition function. We explicitly write only the unknown parameters, namely $[E_n^{c,3}]$ and $|v_n^2(L)|^2$. All other terms at this order, such as $[(E_n^{c,2})^2]$, were already determined in the previous subsection to be the same as in the Nambu-Goto partition function. We then have,

$$\begin{aligned} Z^{ann.} &= \left(\frac{TL}{2\pi R}\right)^{\frac{1}{2}(D-2)} \sum_{n=0,2,4,\cdots}^{\infty} |v_n^0|^2 e^{-R(E_n^{c,0}+E_n^{c,1})}\{\cdots - [E_n^{c,3}]_{ann.}R + \frac{|v_n^2(L)|^2}{|v_n^0|^2} + \cdots\} , \\ Z^{tor.} &= \sqrt{\frac{\pi}{2}}\left(\frac{TL}{R}\right)^{\frac{1}{2}(D-2)} \sum_{n=0}^{\infty} e^{-R(E_n^{c,0}+E_n^{c,1})}\omega_n^{tor.}\{\cdots - [E_n^{c,3}]_{tor.}R + \cdots\} . \end{aligned} \tag{3.17}$$

As in the previous subsections we compute the partition function to six derivative order, given by the following action,

$$S = S_0 + S_2 + S_4 + S_6 . \tag{3.18}$$

The partition function is then,

$$Z(q) = Z_0(q)(1 - \langle S_4\rangle_0 + \frac{1}{2}\langle (S_4)^2\rangle_0 - \langle S_6\rangle_0) . \tag{3.19}$$

Diagrammatically, the c_4 contributions to $\langle S_6\rangle$ are two-loop diagrams similar to the ones of the previous subsection, while the $c_{6,7,8}$ contributions to $\langle S_6\rangle$, and $\langle S_4^2\rangle$, are three-loop diagrams (see Figure 2). We do not compute $\langle S_4^2\rangle$ explicitly since we know from the previous subsection that this contribution is constrained to equal its form in the Nambu-Goto action. In particular, we know it will match all terms (such as $[(E_n^{c,2})^2]$) which we did not write in (3.17), and which do not get contributions from S_6.

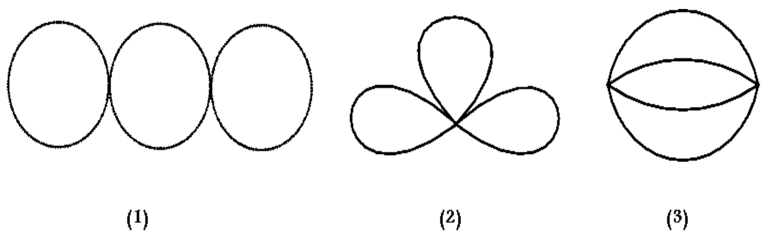

Figure 2: The 3-loop contribution to the partition function. Diagrams (1) and (3) are the two contributions to $\langle S_4^2\rangle$ and diagram (2) is a single vertex diagram appearing in $\langle S_6\rangle$.

The computation of $\langle S_6\rangle$, using the diagrams of figures 1 and 2, gives

$$\begin{aligned} \langle S_6\rangle &= c_4[(D-2)^2 I_3 + 2(D-2)I_4] + c_6[(D-2)^3 I_6 + 6(D-2)^2 I_7 + 8(D-2)I_8] \\ &\quad + c_7[((D-2)^3 + (D-2)^2 + 4(D-2))I_7 + 4(D-1)(D-2)I_8 + (D-2)^2 I_6], \end{aligned} \tag{3.20}$$

where

$$\begin{aligned}
I_3 &= \int d^2\sigma \partial_\alpha \partial^{\alpha'} \partial_\beta \partial^{\beta'} G \partial_\gamma \partial^{\gamma'} G, \quad I_4 = \int d^2\sigma \partial_\alpha \partial_\beta \partial'_\gamma G \partial^\alpha \partial^\beta \partial^{\gamma'} G, \\
I_6 &= \int d^2\sigma \partial_\alpha \partial^{\alpha'} G \partial_\beta \partial^{\beta'} G \partial_\gamma \partial^{\gamma'} G, \\
I_7 &= \int d^2\sigma \partial^\alpha \partial'_\alpha G \partial^\beta \partial'_\gamma G \partial'_\beta \partial^\gamma G, \quad I_8 = \int d^2\sigma \partial^\alpha \partial'_\beta G \partial^\beta \partial'_\gamma G \partial^\gamma \partial'_\alpha G \ .
\end{aligned} \tag{3.21}$$

In appendix A we obtain the following results for the annulus,

$$\begin{aligned}
I_3^{ann.} &= -4\frac{\pi^4 L}{R^5} H_{2,4}(q^{ann.}) = -4\frac{\pi^4 L}{R^5}\left(\frac{4R^5}{15\pi L^5} E_4(\tilde{q}^{ann.}) - \frac{64R^6}{L^6} H_{2,4}(\tilde{q}^{ann.})\right) \ , \\
I_4^{ann.} &= -\frac{1}{2} I_3^{ann.}, \quad I_6^{ann.} = \frac{3\pi^3 L}{2R^5} F(q^{ann.}) = \frac{\pi^3 L}{2R^5}\left[\frac{R^5}{\pi L^5} - \frac{4R^4}{\pi^2 L^4} + \frac{4R^3}{\pi^3 L^3} + O(\tilde{q})\right], \\
I_7^{ann.} &= \frac{1}{2} I_6^{ann.}, \quad I_8^{ann.} = \frac{1}{4} I_6^{ann.},
\end{aligned} \tag{3.22}$$

where $F(q)$ is defined in (A.2). There is an uncertainty $O(\tilde{q})$ in $I_{6,7,8}$ because we computed the $\tilde{q}$ expansion of $F(q)$ numerically, as explained in detail in section A.5.

For the torus we find

$$\begin{aligned}
I_3^{tor.} &= I_4^{tor.} = 0 \ , \\
I_6^{tor.} &= -\frac{1}{L^2 R^2} \ , \ I_7^{tor.} = -\frac{2\pi}{3RL^3} E_2(\tilde{q}^{tor.}) - \frac{\pi^2}{18L^4} E_2^2(\tilde{q}^{tor.}) - \frac{1}{L^2 R^2} \ , \\
I_8^{tor.} &= -\frac{\pi}{RL^3} E_2(\tilde{q}^{tor.}) - \frac{\pi^2}{12L^4} E_2^2(\tilde{q}^{tor.}) - \frac{1}{L^2 R^2} \ .
\end{aligned} \tag{3.23}$$

Using (3.22) and (3.23) in (3.20) we find

$$\begin{aligned}
\langle S_6 \rangle^{ann.} &= -4(D-2)(D-3)c_4\left[\frac{4\pi^3}{15L^4} E_4(\tilde{q}^{ann.}) - \frac{64\pi^4 R}{L^5} H_{2,4}(\tilde{q}^{ann.})\right] \\
&\quad +[(D-2)^3(4c_6+2c_7) + (D-2)^2(12c_6+10c_7) \\
&\quad +(D-2)(8c_6+12c_7)] \times \left[\frac{\pi^2}{8L^4} - \frac{\pi}{2RL^3} + \frac{1}{2R^2L^2} + O(\tilde{q})\right] \ , \\
\langle S_6 \rangle^{tor.} &= -\frac{1}{L^2R^2}(c_6+c_7)[(D-2)^3 + 6(D-2)^2 + 8(D-2)] \\
&\quad -\left(\frac{\pi}{3RL^3} E_2(\tilde{q}^{tor.}) + \frac{\pi^2}{36L^4} E_2^2(\tilde{q}^{tor.})\right)[2c_7(D-2)^3 \\
&\quad +(D-2)^2(12c_6+14c_7) + (D-2)(24c_6+20c_7)] \ .
\end{aligned} \tag{3.24}$$

Putting this result in (3.19) and comparing to (3.17), we are able to find the deviations of the spectrum and overlap functions from the Nambu-Goto (NG) case. We define for convenience: $\Delta E_n = E_n - E_n^{NG}$, $\Delta c_i = c_i - c_i^{NG}$, where E_n^{NG} and c_i^{NG} refer to the NG spectrum and coefficients. By comparing the $\frac{1}{R^2L^2}$ term in the annulus partition function in (3.24) and (3.17), we find the following constraint :

$$\begin{aligned}
0 &= (D-2)^3(4\Delta c_6 + 2\Delta c_7) + (D-2)^2(12\Delta c_6 + 10\Delta c_7) \\
&\quad +(D-2)(8\Delta c_6 + 12\Delta c_7) \ .
\end{aligned} \tag{3.25}$$

This constraint is enough to exclude any correction to the annulus partition function coming from $\Delta c_{6,7}$. The coefficient c_4 is not constrained. To compute the spectrum $[E_n^{c,3}]_{ann.}$ we compare the $\frac{R}{L^5}$ terms. We see that there is a deviation from Nambu-Goto only when $D > 3$ and $c_4 \neq 0$. Since the expansion of $H_{2,4}$ does not contain any constant term, we find that there are deviations from the Nambu-Goto spectrum only for the excited states $n > 0$. Note that this does not teach us about deviations of each state i from the Nambu-Goto form (in its energy or overlap functions), but just that the sum over the states at each energy level should give the results that we stated.

For the torus we find, by comparing the $\frac{1}{L^2R^2}$ and $\frac{1}{L^3R}$ terms and using the consistency of the Nambu-Goto expressions :

$$\begin{aligned}
[\Delta E_n^{c,3}]_{tor.} &= 0, \\
0 &= (\Delta c_6 + \Delta c_7)[(D-2)^3 + 6(D-2)^2 + 8(D-2)], \\
0 &= (D-2)^3(2\Delta c_7) + (D-2)^2(12\Delta c_6 + 14\Delta c_7) \\
&\quad +(D-2)(24\Delta c_6 + 20\Delta c_7) \quad .
\end{aligned} \tag{3.26}$$

On the torus, the partition function is exactly the Nambu-Goto partition function at this order, and there are no contributions at all coming from $\Delta c_{4,6,7}$. This result is different from what we got for the annulus, but there is no contradiction since in the torus there are contributions from all states, while in the annulus only some of the states contribute. Therefore our results imply a relation between the sums of the corrections to the energies of the states with the different possible momenta at each level. For instance, at the level $n = 2$, there is one possible value $(N_L, N_R) = (1, 1)$ for the annulus and 3 possible values for the torus $(N_L, N_R) = (0, 2), (2, 0), (1, 1)$. Thus, at this level we have the relation $\Delta E_{(1,1)} = -2\Delta E_{(2,0)} = -2\Delta E_{(0,2)}$. If $D = 3$ or $c_4 = 0$ then we obtain from the annulus the additional relation $\Delta E_{(1,1)} = 0$, and the corrections to the energies of the $(2, 0)$,$(0, 2)$ states also sum to zero.

The two constraints we find on $c_{6,7}$ from the torus are linearly independent of each other, but not of the annulus constraint, so there is a consistent solution. For $D = 3$, there is only one independent term which is constrained to have the Nambu-Goto coefficient. For $D > 3$ the general solution to the two constraints turns out to be independent of D,

$$c_6 = \frac{1}{16T^2}, \quad c_7 = -\frac{1}{8T^2}, \tag{3.27}$$

so that these coefficients also exactly agree with their Nambu-Goto values.

We summarize our results at this order:

For $D = 3$ there is a single parameter in the effective action, c_6 which is constrained to be the Nambu-Goto coefficient. Thus, the effective action to this order is the same as the Nambu-Goto action, and all energy levels should agree with the Nambu-Goto levels up to order $1/L^5$.

For $D > 3$, there are three independent terms : c_4 which is unconstrained, and $c_{6,7}$ which have two constraints and agree with their Nambu-Goto values. The annulus partition function is generally affected by c_4, but the ground state energy has no deviations from

Nambu-Goto at this order. The torus partition function at this order is not affected by c_4, and it is always equal to the Nambu-Goto partition function.

4. Superstrings in confining backgrounds

String theory in flat space-time is described, in the conformal gauge, by a free worldsheet theory of massless degrees of freedom (d.o.f), corresponding to the worldsheet fluctuations. In the general, non-flat, case there are background fields with non-trivial vacuum expectation values. These fields couple to the worldsheet d.o.f and create interaction terms, so that the worldsheet theory is no longer free.

In this section we will derive the worldsheet action for an infinite confining string, in a specific class of holographic backgrounds. In these backgrounds, the confining string sits at the minimal value of a radial coordinate. As we will see, in the static gauge, the worldsheet action in such a configuration will contain massive and massless modes. We will then be able, in later sections, to define an effective action which includes the massless fields only. This action can be considered on different topologies, as in the previous section.

We begin with a description of the possible background fields and heuristically describe their coupling to the worldsheet theory. This discussion should make clear how general our analysis is, and which backgrounds it fails to describe (examples of backgrounds which are included in our analysis will be presented in the next section). Then, we present the superstring action in these backgrounds, which was derived in [34, 35], and we derive its Feynman rules. To avoid confusion, we summarize our notations in appendix B.

4.1 The backgrounds

The class of confining backgrounds which we consider (a sub-class of the general confining backgrounds discussed in [36]) contains a cycle $S^{p'}$ (with the topology of a sphere) that vanishes smoothly at the minimal value of the radial coordinate, ρ_0. At ρ_0 the warp factor, $f(\rho)$, is minimal and so the string is forced to sit at this point. We can write the radial direction and the $S^{p'}$ coordinates together in Cartesian coordinates, $\rho \in \mathbf{R}^{p'+1}$, and expand the metric around the minimal point ρ_0. Up to the order we will need in $\Delta\rho = \rho - \rho_0$, the metric of the X and ρ coordinates is given by (in the string frame)

$$\frac{ds^2}{2\pi\alpha'} = f(\rho)dX^2 + d\rho^2 = f(\rho_0)\left(1 + \frac{f''(\rho_0)}{2f(\rho_0)}\Delta\rho^2\right)dX^2 + d\rho^2. \tag{4.1}$$

The string is stretched along the X directions, where the field theory lives. The effective string tension in α' units is $T = f(\rho_0)$. The coefficient of the $\Delta\rho^2 dX^2$ term is proportional to the curvature $\mathcal{R}$ at the minimal point ,$\frac{f''(\rho_0)}{f(\rho_0)} \propto \frac{\mathcal{R}}{T}$, which is positive. The metric should be smooth, so the warp factor is a function of $\Delta\rho^2$ and there are no odd powers appearing in the expansion.

Expanding the metric perturbatively near the minimal point is valid when the curvature is small compared with the string tension, $\mathcal{R} << T$. This limit often corresponds to a large 't Hooft coupling limit in the dual field theories. In the static gauge, the quadratic

part of the warp factor will create worldsheet masses (and interaction terms) for the ρ coordinates. This is rather intuitive, as the curvature suppresses excitations in the ρ directions, and in the language of the worldsheet theory causes these fields to become heavy.

There is usually also another compact p dimensional subspace, with a radius which scales like a power of N. As discussed in section 2, we will not take these coordinates into consideration. However, these compact subspaces are stabilized by some flux which they carry, and this flux will appear in our computations. Since the geometry is not flat, it is convenient to choose for these coordinates an appropriate vielbein ($e^a \cdot e^b = \delta^{ab}$) to work with. In this frame the volume form is always proportional to the antisymmetric tensor, and is not coordinate dependent.

The metric which we will use is thus (rescaling the X coordinates, renaming the ρ coordinates Y, and renaming the curvature term m_B^2)

$$\frac{ds^2}{2\pi\alpha'} = (1 + \frac{m_B^2}{2T} Y_B^2) \sum_{\xi=0}^{D-1} dX^\xi dX_\xi + \sum_{B=D}^{D+N_B-1} dY_B dY_B + \frac{1}{2\pi\alpha'} \sum_{a=D+N_B}^{9} de^a de^a. \quad (4.2)$$

Here $N_B = p' + 1$ is the number of massive scalars on the worldsheet. The ten space-time coordinates, Z^μ ($\mu = 0, \cdots, 9$), include a $\mathbf{R}^D$ part, which is spanned by the X coordinates and is multiplied by the warp factor, and also the Y and e^a coordinates. Our X and Y coordinates are dimensionless. In our computation we will integrate out at one-loop order the Y coordinates and the massive fermions. This means that in our action we only need to keep terms up to quadratic order in these fields.

We choose the static gauge fixing $X^\alpha = \sqrt{T}\sigma^\alpha$ ($\alpha = 0, 1$), where T is the string tension. In this gauge m_B^2 become the masses of the radial coordinates Y_B, where B is an index running over all these fields. Note that in this gauge, the range of the dimensionless X on (say) the torus is $0 \leq X^0 \leq L\sqrt{T}$, $0 \leq X^1 \leq R\sqrt{T}$.

The second possible bosonic background is the NS-NS 2-form field. We assume this field is not polarized in the $\mathbf{R}^D$ directions, and therefore has no interaction with the X coordinates to one loop order. For $D > 3$, this assumption follows from Lorentz invariance.

The background generally includes also various R-R field strengths which couple to the worldsheet fermions through the covariant derivative including terms proportional to $\bar{\Theta} F_{\mu_1 \cdots \mu_p} \Gamma^{\mu_1 \cdots \mu_p} \Theta$. For each p-form we define $\tilde{\Gamma}_p \equiv \frac{1}{8p!} e^\phi F_{\mu_1 \cdots \mu_p} \Gamma^{\mu_1 \cdots \mu_p}$ (this has units of mass in our conventions). We work under the assumption that $\tilde{\Gamma}_p$ can be expressed with gamma matrices polarized orthogonally to the $\mathbf{R}^D$ directions. There may be non-zero background fields in these directions, such as the self-dual 5-form in the original AdS/CFT correspondence [15], which is polarized in all ten dimensions. However, because of Lorentz invariance in the $\mathbf{R}^D$ directions, $\tilde{\Gamma}_p$ is a sum over gamma matrices which either contain all D directions (and possibly other directions), or none. Therefore, the gamma matrices which are polarized in the flat directions can be expressed as gamma matrices polarized in orthogonal directions using the chirality operator; e.g. $\tilde{\Gamma}_5 = F_{01234}\Gamma^{01234} + F_{56789}\Gamma^{56789} = (F_{01234}\Gamma^{11} + F_{56789})\Gamma^{56789}$.

To conclude, our analysis will include all the possible p-form backgrounds which appear in IIA/B superstring theories, apart from the B-field which, if polarized orthogonally to

the field theory dimensions, does not couple directly to the X scalars, and it can only give interaction terms which do not contribute in our one-loop computation.

There may be more then one background p-form field present, and their sum generates a mass and interaction terms for the fermions. For simplicity we will start with the case of a single p-form background. As we will discover, the fermionic action is identical for all p-forms which we consider (up to some sign differences). At this point, the addition of several forms will be trivial. In our convention $\tilde{\Gamma}_p$ is a real matrix that has the following symmetry and commutation properties with Γ^i $(i = 0, \cdots, D-1)$

$$\begin{array}{ccc} & \tilde{\Gamma}_p^T = \tilde{\Gamma}_p & \tilde{\Gamma}_p^T = -\tilde{\Gamma}_p \\ IIA\ ([\tilde{\Gamma}_p, \Gamma^i] = 0) & p = 4 & p = 2 \\ IIB\ (\{\tilde{\Gamma}_p, \Gamma^i\} = 0) & p = 1, 5 & p = 3 \end{array} \tag{4.3}$$

Each $\tilde{\Gamma}_p$ matrix is symmetric (skew-symmetric), and therefore by itself diagonalizable (block diagonalizable). As we will see there are other matrices which multiply $\tilde{\Gamma}_p$, so the final mass matrix is always skew-symmetric, and we can bring it to a block diagonalized form, which is the standard form for Dirac mass terms. For the total mass matrix to be brought into this form, we need all the matrices which originate in different fluxes to commute with each other. This occurs in all the examples which we discuss, and should be the case in any well-defined worldsheet theory. We stress that although the fermion and scalar masses come from different background fields, they are always related through the background equations of motion to give the sum rule (2.16). This is necessary for our results to be finite, as we will see in section 6. We will discuss separately the type IIA and type IIB cases, and show that they lead to the same action in the backgrounds we consider.

4.2 Type IIA action

We begin with the type IIA action in its Polyakov form, in the Green-Schwarz (GS) formalism, to second order in fermions [34, 35, 37]:

$$\begin{aligned} S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \{ & \sqrt{-h} h^{\alpha\beta} (\partial_\alpha Z^\mu \partial_\beta Z_\mu - 2i \partial_\alpha Z^\mu \bar{\Theta} \Gamma_\mu D_\beta \Theta) \\ & + 2i \epsilon^{\alpha\beta} \partial_\alpha Z^\mu \bar{\Theta} \Gamma^{11} \Gamma_\mu D_\beta \Theta \}, \\ D_\alpha \equiv \partial_\alpha + \sum_p & \partial_\alpha Z^\mu \tilde{\Gamma}_p \Gamma_\mu Q_p, \quad \{Q_2 = \Gamma^{11}\ ,\ Q_4 = I\} \quad . \end{aligned} \tag{4.4}$$

Here the fields Z^μ are contracted with the metric $g_{\mu\nu}$ written above (4.2). Θ is a space-time Majorana fermion with 32 real degrees of freedom off-shell (the Majorana condition is taken such that the fermions are real variables). The gamma matrices obey the general relation $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$. The worldsheet directions are $\alpha, \beta = 0, 1$. Apart from diffeomorphism and Weyl invariance, the action is kappa symmetric and reduces to the familiar GS action in flat space-time. The matrices Q_p are needed when we write the action in this compact form, without an explicit summation over two Weyl fermions as in [38]. These matrices ensure that the matrix sitting between the two fermions is antisymmetric.

Since we are interested in describing the low-energy effective action on a long string, the convenient gauge to work in is the static gauge $X^\alpha = \sigma^\alpha\sqrt{T}$ $(\alpha = 0,1)$ [5]. This is useful as we have a dimensionless parameter to expand in, which is $k^i/\sqrt{T}$, where k^i are the momenta of the fields on the string. After this gauge fixing, we cannot completely gauge away the worldsheet metric $h_{\alpha\beta}$. We therefore set the metric to its classical value, using the equations of motion, and expand around this solution:

$$\begin{aligned} K_{\alpha\beta} &\equiv \partial_\alpha Z^\mu \partial_\beta Z_\mu - i(\partial_\alpha Z^\mu \bar{\Theta}\Gamma_\mu D_\beta \Theta + \alpha \leftrightarrow \beta) \ , \\ h_{\alpha\beta}\sqrt{-K} &= \sqrt{-h} K_{\alpha\beta} \ , \\ h &\equiv \det(h_{\alpha\beta}) \ , \ K \equiv \det(K_{\alpha\beta}) \ . \end{aligned} \tag{4.5}$$

Integrating out the metric classically we obtain the following Nambu-Goto like action

$$\begin{aligned} S_{NG} &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\{\sqrt{-K} + S_2\} \ , \\ S_2 &= i\epsilon^{\alpha\beta}\partial_\alpha Z^\mu \bar{\Theta}\Gamma^{11}\Gamma_\mu D_\beta\Theta \ . \end{aligned} \tag{4.6}$$

As mentioned earlier, we will not consider effects from loops of the massless fields in the theory, and so we will ignore the worldsheet metric fluctuations. In principle we still need to fix the Weyl gauge symmetry, however this symmetry acts only on the metric $h_{\alpha\beta}$ so this will not affect our computation. Note that the metric does not have a kinetic term, and therefore will not contribute to the Lüscher term.

We split our Majorana fermion into two space-time Weyl-Majorana fermions:

$$\Theta = \Theta^1 + \Theta^2 \quad , \quad \Gamma^{11}\Theta^I = (-1)^{I+1}\Theta^I, \quad (I = 1,2) \quad . \tag{4.7}$$

We write the gamma matrices as a product of gamma matrices in the 2 dimensional worldsheet directions and the transverse 8 directions. In the following, $\rho_\alpha(\gamma_i)$ are 2(8) dimensional gamma matrices in flat space (see appendix B for more details), and then we have

$$\begin{aligned} \Gamma_\alpha &= \sqrt{2\pi\alpha'}\rho_\alpha \otimes I \quad , \quad \Gamma_i = \sqrt{2\pi\alpha'}\rho \otimes \gamma_i \quad , \\ \tilde{\Gamma}_p &\equiv \frac{1}{\sqrt{2\pi\alpha'}}\rho^p \otimes \tilde{\gamma}_p, \\ \rho &\equiv \rho_0\rho_1 \quad , \quad \gamma^c \equiv \gamma^2\cdots\gamma^9 \quad , \quad \Gamma^{11} \equiv \rho \otimes \gamma^c \quad . \end{aligned} \tag{4.8}$$

Here ρ is the worldsheet chirality operator, γ^c is the chirality operator in the transverse 8 dimensions, and Γ^{11} is the 10 dimensional chirality operator. In (4.8) we wrote explicitly the vielbein $e^{a(i)}_{b(j)} \propto \delta_{a(i),b(j)}\sqrt{2\pi\alpha'}$ (to leading order in the radial variables Y). We fix the kappa symmetry by identifying the worldsheet chirality of the fermions with their space-time chirality. This means that both fermions have positive 8 dimensional chirality in the transverse directions. In our basis of gamma matrices the kappa fixing then takes the form

$$\rho\Theta^I = \Gamma^{11}\Theta^I \quad , \quad \gamma^c\Theta^I = \Theta^I \quad . \tag{4.9}$$

[5]Note that in this gauge we still need to take into account the equation of motion of X^α which is not automatically satisfied, and should be viewed as a constraint. At leading order the constraint follows from the other equations of motion, so the leading non-trivial constraint on physical states arises at 5-derivative order and involves four fields. It would be interesting to find a different formalism in which the constraints are automatically satisfied.

This eliminates half of the degrees of freedom carried by the fermions, and our space-time spinors now have a worldsheet spinor index and an 8-dimensional space-time spinor index. This kappa fixing is not entirely arbitrary, as we shall now explain.

Our action (4.6) is invariant under local fermionic transformations which are generalizations of the flat-space kappa symmetry transformation presented in [38],

$$\begin{aligned}
\delta_k\Theta &= 2i\Gamma\cdot\Pi_\alpha P^{\alpha\beta}k_\beta \quad , \quad \delta_k X^\mu = i\bar{\Theta}\Gamma^\mu\delta_k\Theta \\
P^{\alpha\beta} &\equiv \frac{1}{2}(K^{\alpha\beta} - \frac{\epsilon^{\alpha\beta}\Gamma^{11}}{\sqrt{-K}}) \quad , \quad \Pi^\mu_\alpha \equiv \partial_\alpha X^\mu - i\bar{\Theta}\Gamma^\mu\partial_\alpha\Theta \ .
\end{aligned} \tag{4.10}$$

Here k_α is a Majorana fermion in space-time and $P^{\alpha\beta}$ is a projector operator, affecting only half of the d.o.f in Θ. When we split our Majorana fermions in (4.7), each fermion $\Theta^{1,2}$ is only affected by the part of k which has the opposite space-time chirality. We then define $k = k^1 + k^2$, where $\Gamma^{11}k^i = (-1)^i k^i$. In the static gauge the projection operator $P^{\alpha\beta}$ becomes a worldsheet chirality projector,

$$\delta_k\Theta^1 = -4i\Gamma_- k^1 + O(Z^\mu) \quad , \quad \delta_k\Theta^2 = -4i\Gamma_+ k^2 + O(Z^\mu) \ , \tag{4.11}$$

where $\Gamma_\pm = \frac{1}{2}(\Gamma_0 \pm \Gamma_1)$ refer to the lightcone coordinates. Here $O(Z^\mu)$ refers to corrections which involve other fields. A simple gauge fixing which completely fixes our gauge freedom is then [35]

$$\Gamma^-\Theta^1 = 0 \quad , \quad \Gamma^+\Theta^2 = 0 \ . \tag{4.12}$$

One should make sure that the Fadeev-Popov determinant of our full gauge fixing, namely the kappa symmetry and diffeomorphism gauge fixing, does not vanish. This determinant will contain a kinetic term for k_i, which will become ghost fields, and the $O(Z^\mu)$ term in the kappa transformation will produce interaction terms between the ghosts and the other coordinates. We have verified the consistency of our gauge-fixing, but we will not describe this in detail here.

After taking the static gauge the X^α no longer appear in the action, which involves (up to terms which do not contribute in our one-loop computation)

$$\begin{aligned}
K_{\alpha\beta} &= -\delta_{\alpha,o}\delta_{\beta,0}2\pi\alpha' T + \delta_{\alpha,1}\delta_{\beta,1}2\pi\alpha' T + \partial_\alpha X^i\partial_\beta X_i + \partial_\alpha Y^B\partial_\beta Y_B \\
&+ i\sqrt{2\pi\alpha' T}(\Theta^1\rho_1(\rho_\alpha\partial_\beta + \rho_\beta\partial_\alpha)\Theta^1 - \Theta^2\rho_1(\rho_\alpha\partial_\beta + \rho_\beta\partial_\alpha)\Theta^2) \\
&\pm i\sqrt{2\pi\alpha' T}[\Theta^1\rho_1\{\rho_\alpha, \rho_\beta\}\tilde{\gamma}_p\Theta^2 - \Theta^2\rho_1\{\rho_\alpha, \rho_\beta\}\tilde{\gamma}_p^T\Theta^1] \\
&\pm 2i\sqrt{2\pi\alpha'}\partial_\alpha X^i\partial_\beta X_i[\Theta^1\rho_1\tilde{\gamma}_p\Theta^2 - \Theta^2\rho_1\tilde{\gamma}_p^T\Theta^1], \\
S_2 &= 2i\sqrt{2\pi\alpha' T}(\Theta^1\partial_+\Theta^1 + \Theta^2\partial_-\Theta^2) \pm 2i\sqrt{2\pi\alpha' T}(\Theta^1\tilde{\gamma}_p\Theta^2 - \Theta^2\tilde{\gamma}_p^T\Theta^1) \\
&\pm \frac{i}{2}\sqrt{2\pi\alpha'}\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j(\Theta^1[\gamma_i, \gamma_j]\tilde{\gamma}_p\Theta^2 + \Theta^2[\gamma_i, \gamma_j]\tilde{\gamma}_p^T\Theta^1) \quad .
\end{aligned} \tag{4.13}$$

The upper(lower) sign relates to the 4(2)-form background, and the relative minus sign between the two backgrounds can be swallowed into $\tilde{\gamma}_p$. Since our final results will only depend on physical quantities, such as the masses squared of the fermions (which are proportional to $\tilde{\gamma}_p^2$), this will not make a difference. Thus, we can see that the two different

possible type IIA backgrounds give the same form to the action. When deriving (4.13), we used the worldsheet chirality of the fermions to express ρ^0 as ρ_1 which is a positive matrix. One should notice that there are no interactions linear in X^i. This is because of our gauge choice for kappa symmetry fixing and the assumptions on $\tilde{\Gamma}_p$ which includes no gamma matrices in the worldsheet directions. Notice that fermions of opposite chirality only mix through mass terms proportional to $\tilde{\gamma}_p$, and so this mixing vanishes in flat space as it should.

4.3 Type IIB action

Here we start directly with the NG like action for the type IIB case [34, 35], given by (4.6) with

$$\begin{aligned} K_{\alpha\beta} &\equiv \partial_\alpha Z^\mu \partial_\beta Z_\mu - i(\partial_\alpha Z^\mu \bar{\Theta}^I \Gamma_\mu D^{IJ}_\beta \Theta^J + \alpha \leftrightarrow \beta), \quad (I,J=1,2), \\ S_2 &= i\epsilon^{\alpha\beta}\partial_\alpha Z^\mu \bar{\Theta}^I \rho^{IJ}\Gamma^{11}\Gamma_\mu D^{JK}_\beta \Theta^K \ , \\ D^{IJ}_\alpha &\equiv \partial_\alpha \delta^{IJ} + \partial_\alpha Z^\mu \tilde{\Gamma}_p \Gamma_\mu Q^{IJ}_p, \quad \{Q^{IJ}_{p=1,5} = \rho^{IJ}_0, \quad Q^{IJ}_{p=3} = -\rho^{IJ}_1\}. \end{aligned} \tag{4.14}$$

In the type IIB case we have two fermions $\Theta^{1,2}$, both with positive space-time chirality. One technical difference between the type IIA and type IIB actions, is that the first is initially written for a single Majorana fermion, while the latter is written in terms of two fermions. This difference is because in the type IIA case we can use the space-time chirality operator to distinguish the right and left movers and so to write down the correct interactions. After gauge-fixing and using the metric (4.2), the determinant K and the topological part S_2 are the same in the type IIB case as in (4.13), where the upper sign refers to the 1,5-form cases and the lower sign to the 3-form case. The kappa symmetry fixing is restricted to be the same as in the type IIA case, so that here the worldsheet chirality of the fermions is not the same as their space-time chirality, but as their 8-dimensional chirality :

$$\Gamma^{11}\Theta^I = \Theta^I, \quad \rho\Theta^I = (-1)^{I+1}\Theta^I, \quad \gamma^c\Theta^I = (-1)^{I+1}\Theta^I. \tag{4.15}$$

Again, we see that the action is the same for all flux sectors.

When we have several background fields, we will still have the same action, but we should replace $\tilde{\gamma}_p$ by the sum $\sum_p \tilde{\gamma}_p$. As we claimed before, this sum of matrices can always be block diagonalized. Both for type IIA and for type IIB we can re-express it in the following way, in terms of projection operators $\tilde{\gamma}_F$ on mass eigenstates:

$$\sum_p \tilde{\gamma}_p = \sum_F \frac{m_F}{2\sqrt{T}}\tilde{\gamma}_F \quad , \quad \mathrm{tr}[\tilde{\gamma}^T_F \tilde{\gamma}_F] = \frac{1}{8}\mathrm{tr}[1] = 2, \quad (\tilde{\gamma}^T_F \tilde{\gamma}_F)^2 = \tilde{\gamma}^T_F \tilde{\gamma}_F \quad . \tag{4.16}$$

These matrices obey the commutation relation $[\tilde{\gamma}_F, \gamma^c] = 0$ for the type IIA theory, and the anti-commutation relation $\{\tilde{\gamma}_F, \gamma^c\} = 0$ for type IIB. The sum $\sum_F$ is over all massive fermions. $\tilde{\gamma}^T_F \tilde{\gamma}_F$ is a projection operator in the 16-dimensional spinor space, projecting onto one on-shell fermionic d.o.f. In this description we have eight worldsheet Dirac fermions with (generally) different masses. An example of such a decomposition is given in section 5.3 where the Maldacena-Nuñez R-R background is discussed.

4.4 The Nambu-Goto determinant

We now carry on with a derivative expansion of the action (4.6). In the rest of the section we will use the Einstein summation rule for the worldsheet coordinates. We incorporate the space-time metric explicitly, and use $A \cdot A = \sum_a A_a A_a$, with the relevant indices for X and Y coordinates. Our NG-like action contains a square root of the determinant $K = K_{00}K_{11} - K_{01}K_{10}$. This can be consistently expanded in powers of derivatives over the tension, $\frac{k}{\sqrt{T}} \ll 1$. We will perform this expansion up to sixth order in derivatives and up to second order in the massive fields Y and Θ [6]. This will be enough to compute the one-loop effective action of the X's by integrating out the massive fields. We rescale the fermionic fields $\Theta \to (\frac{2\pi\alpha'}{64T})^{\frac{1}{4}}\Theta$. The spinors Θ^1, Θ^2 are Weyl spinors, and have only one d.o.f on the worldsheet ($\Theta^1 = \begin{pmatrix} \theta^1 \\ 0 \end{pmatrix}$, $\Theta^2 = \begin{pmatrix} 0 \\ \theta^2 \end{pmatrix}$). From here on we write our action in terms of θ^1 and θ^2, which are worldsheet scalars. The elements of K are given by

$$
\begin{aligned}
\frac{K_{00}}{2\pi\alpha'} &= (-T + \partial_0 X \cdot \partial_0 X)(1 + \frac{m_B^2}{2T} Y_B^2) + \partial_0 Y \cdot \partial_0 Y \\
&\quad - \frac{i}{4}(\theta^1 \partial_0 \theta^1 + \theta^2 \partial_0 \theta^2) + \frac{i m_F}{8}(\theta^1 \tilde{\gamma}_F \theta^2 - \theta^2 \tilde{\gamma}_F^T \theta^1)(\frac{1}{T}\partial_0 X \cdot \partial_0 X - 1), \\
\frac{K_{11}}{2\pi\alpha'} &= (T + \partial_1 X \cdot \partial_1 X)(1 + \frac{m_B^2}{2T} Y_B^2) + \partial_1 Y \cdot \partial_1 Y \\
&\quad + \frac{i}{4}(\theta^1 \partial_1 \theta^1 - \theta^2 \partial_1 \theta^2) + \frac{i m_F}{8}(\theta^1 \tilde{\gamma}_F \theta^2 - \theta^2 \tilde{\gamma}_F^T \theta^1)(\frac{1}{T}\partial_1 X \cdot \partial_1 X + 1), \\
\frac{K_{01}}{2\pi\alpha'} &= \partial_0 X \cdot \partial_1 X (1 + \frac{m_B^2}{2T} Y_B^2) + \partial_0 Y \cdot \partial_1 Y + \frac{i}{4}(\theta^1 \partial_- \theta^1 + \theta^2 \partial_+ \theta^2) \\
&\quad + \frac{i m_F}{8T}\partial_0 X \cdot \partial_1 X (\theta^1 \tilde{\gamma}_F \theta^2 - \theta^2 \tilde{\gamma}_F^T \theta^1),
\end{aligned}
\tag{4.17}
$$

and its determinant (to the order we need) by

$$
\begin{aligned}
\frac{-K}{(2\pi\alpha' T)^2} &= 1 + \frac{1}{T}\partial_\alpha X \cdot \partial^\alpha X + \frac{1}{T}\partial_\alpha Y \cdot \partial^\alpha Y + \frac{m_B^2}{T} Y_B^2 + \frac{i}{2T}(\theta^1 \partial_+ \theta^1 + \theta^2 \partial_- \theta^2) \\
&+ \frac{i m_F}{4T}(\theta^1 \tilde{\gamma}_F \theta^2 - \theta^2 \tilde{\gamma}_F^T \theta^1) \\
&+ \frac{1}{T^2}\partial_\alpha X \cdot \partial^\alpha X \{\partial_\beta Y \cdot \partial^\beta Y + m_B^2 Y_B^2 + \frac{i}{2}(\theta^1 \partial_+ \theta^1 + \theta^2 \partial_- \theta^2) + \frac{i m_F}{4}(\theta^1 \tilde{\gamma}_F \theta^2 - \theta^2 \tilde{\gamma}_F^T \theta^1)\} \\
&- \frac{1}{T^2}\partial_\alpha X \cdot \partial_\beta X \partial^\alpha Y \cdot \partial^\beta Y - \frac{i}{2T^2}\partial_\alpha X \cdot \partial_+ X \theta^1 \partial^\alpha \theta^1 - \frac{i}{2T^2}\partial_\alpha X \cdot \partial_- X \theta^2 \partial^\alpha \theta^2 \\
&+ \frac{1}{T^3}((\partial^\alpha X \cdot \partial_\alpha X)^2 - \partial_\alpha X \cdot \partial_\beta X \partial^\alpha X \cdot \partial^\beta X)\{\frac{T}{2} + \frac{m_B^2}{2} Y_B^2 + \frac{i m_F}{8}(\theta^1 \tilde{\gamma}_F \theta^2 - \theta^2 \tilde{\gamma}_F^T \theta^1)\}\ .
\end{aligned}
\tag{4.18}
$$

Already at this stage we can see that in the static gauge, we have a canonical kinetic term for the fermions, and part of the interactions became mass terms for the fermions and scalars.

[6] Actually we will not require in this paper the six-derivative terms with fermion couplings, so we will not write them down.

Our first step will be to expand the square root in the action (4.6) in powers of $-\frac{K}{(2\pi\alpha' T)^2}-1$, to the order we are interested in:

$$\begin{aligned} S = -T\int d^2\sigma\{1 &+ \frac{1}{2}(\frac{-K}{(2\pi\alpha' T)^2}-1) - \frac{1}{8}(\frac{-K}{(2\pi\alpha' T)^2}-1)^2 + \frac{1}{16}(\frac{-K}{(2\pi\alpha' T)^2}-1)^3 \\ &- \frac{5}{128}(\frac{-K}{(2\pi\alpha' T)^2}-1)^4 + \frac{1}{2\pi\alpha' T}S_2\} \quad . \end{aligned} \tag{4.19}$$

Keeping only terms we are interested in, the powers in S are given by:

$$\begin{aligned} &(\frac{-K}{(2\pi\alpha' T)^2}-1)^2 = \frac{2}{T^2}\partial_\alpha X\cdot\partial^\alpha X\{\partial_\beta Y\cdot\partial^\beta Y + m_B^2 Y_B^2 \\ &-\frac{i}{2}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2) + \frac{im_F}{4}(\theta^1\tilde\gamma_F\theta^2-\theta^2\tilde\gamma_F^T\theta^1)\} - \frac{2}{T^3}\partial^\gamma X\cdot\partial_\gamma X\partial_\alpha X\cdot\partial_\beta X\partial^\alpha Y\cdot\partial^\beta Y \\ &-\frac{1}{T^3}\partial^\gamma X\cdot\partial_\gamma X\{\partial_\alpha X\cdot\partial_+ X\theta^1\partial^\alpha\theta^1 + \partial_\alpha X\cdot\partial_- X\theta^2\partial^\alpha\theta^2\} + \frac{1}{T^2}(\partial^\alpha X\cdot\partial_\alpha X)^2 \\ &+\frac{1}{T^3}(3(\partial^\alpha X\cdot\partial_\alpha X)^2 - \partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\beta X)\{\partial_\gamma Y\cdot\partial^\gamma Y + m_B^2 Y_B^2 \\ &+\frac{i}{2}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2) + \frac{im_F}{4}(\theta^1\tilde\gamma_F\theta^2-\theta^2\tilde\gamma_F^T\theta^1)\} \\ &+\frac{1}{T^4}((\partial_\alpha X\cdot\partial^\alpha X)^3 - \partial_\gamma X\cdot\partial^\gamma X\partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\beta X)(T+\partial_\delta Y\cdot\partial^\delta Y + 2m_B^2 Y_B^2) \\ &-\frac{1}{T^4}((\partial_\alpha X\cdot\partial^\alpha X)^2 - \partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\alpha X)\partial_\gamma X\cdot\partial_\delta X\partial^\gamma Y\cdot\partial^\delta Y, \end{aligned} \tag{4.20}$$

$$\begin{aligned} &(\frac{-K}{(2\pi\alpha' T)^2}-1)^3 = \frac{3}{T^3}(\partial_\alpha X\cdot\partial^\alpha X)^2\{\partial_\beta Y\cdot\partial^\beta Y + m_B^2 Y_B^2 \\ &\qquad\qquad +\frac{i}{2}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2) + \frac{im_F}{4}(\theta^1\tilde\gamma_F\theta^2-\theta^2\tilde\gamma_F^T\theta^1)\} \\ &+\frac{1}{T^4}(\partial_\alpha X\cdot\partial^\alpha X)^3(T+6\partial_\beta Y\cdot\partial^\beta Y + 6m_B^2 Y_B^2) - \frac{3}{T^4}(\partial^\alpha X\cdot\partial_\alpha X)^2\partial_\beta X\cdot\partial_\gamma X\partial^\beta Y\cdot\partial^\gamma Y \\ &-\frac{3}{T^4}(\partial^\gamma X\cdot\partial_\gamma X\partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\beta X)(\partial_\delta Y\cdot\partial^\delta Y + m_B^2 Y_B^2), \end{aligned} \tag{4.21}$$

and

$$(\frac{-K}{(2\pi\alpha' T)^2}-1)^4 = \frac{4}{T^4}(\partial_\alpha X\cdot\partial^\alpha X)^3\{\partial_\beta Y\cdot\partial^\beta Y + m_B^2 Y_B^2\} \quad . \tag{4.22}$$

The full action to order $O((\partial X)^6 Y^2, (\partial X)^4\theta^2)$ is then

$$
\begin{aligned}
S = & -\int d^2\sigma\{T+\frac{1}{2}\partial_\alpha X\cdot\partial^\alpha X+\frac{1}{2}\partial_\alpha Y\cdot\partial^\alpha Y+\frac{1}{2}m_B^2Y_B^2 \\
& +\frac{i}{2}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2)+\frac{im_F}{4}(\theta^1\tilde{\gamma}_F\theta^2-\theta^2\tilde{\gamma}_F^T\theta^1) \\
& +\frac{1}{4T}\partial_\alpha X\cdot\partial^\alpha X[\partial_\beta Y\cdot\partial^\beta Y+m_B^2Y_B^2+\frac{i}{2}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2)+\frac{im_F}{4}(\theta^1\tilde{\gamma}_F\theta^2-\theta^2\tilde{\gamma}_F^T\theta^1)] \\
& -\frac{1}{2T}\partial_\alpha X\cdot\partial_\beta X\partial^\alpha Y\cdot\partial^\beta Y-\frac{i}{4T}\partial_\alpha X\cdot\partial_+X\theta^1\partial^\alpha\theta^1-\frac{i}{4T}\partial_\alpha X\cdot\partial_-X\theta^2\partial^\alpha\theta^2 \\
& +\frac{1}{4T^2}\partial^\gamma X\cdot\partial_\gamma X\partial_\alpha X\cdot\partial_\beta X\partial^\alpha Y\cdot\partial^\beta Y \\
& +\frac{i}{8T^2}\partial^\gamma X\cdot\partial_\gamma X[\partial_\alpha X\cdot\partial_+X\theta^1\partial^\alpha\theta^1+\partial_\alpha X\cdot\partial_-X\theta^2\partial^\alpha\theta^2] \\
& +\frac{1}{T^2}(\partial^\alpha X\cdot\partial_\alpha X)^2[\frac{T}{8}-\frac{3}{16}\partial_\beta Y\cdot\partial^\beta Y+\frac{1}{16}m_B^2Y_B^2-\frac{3i}{32}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2) \\
& +\frac{im_F}{64}(\theta^1\tilde{\gamma}_F\theta^2-\theta^2\tilde{\gamma}_F^T\theta^1)] \\
& +\frac{1}{T^2}(\partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\beta X)[-\frac{T}{4}+\frac{1}{8}\partial_\gamma Y\cdot\partial^\gamma Y-\frac{1}{8}m_B^2Y_B^2+\frac{i}{16}(\theta^1\partial_+\theta^1+\theta^2\partial_-\theta^2) \\
& -\frac{im_F}{32}(\theta^1\tilde{\gamma}_F\theta^2-\theta^2\tilde{\gamma}_F^T\theta^1)]+\frac{im_F}{32T}\epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j(\theta^1[\gamma_i,\gamma_j]\tilde{\gamma}_F\theta^2+\theta^2[\gamma_i,\gamma_j]\tilde{\gamma}_F^T\theta^1) \\
& +\frac{1}{32T^3}(\partial_\alpha X\cdot\partial^\alpha X)^3(-2T+3\partial^\beta Y\cdot\partial_\beta Y-m_B^2Y_B^2)-\frac{1}{16T^3}(\partial_\alpha X\cdot\partial^\alpha X)^2\partial_\beta X\cdot\partial_\gamma X\partial^\beta Y\cdot\partial^\gamma Y \\
& +\frac{1}{16T^3}\partial_\alpha X\cdot\partial^\alpha X\partial_\beta X\cdot\partial_\gamma X\partial^\beta X\cdot\partial^\gamma X(2T-\partial_\delta Y\partial^\delta Y+m_B^2Y_B^2) \\
& -\frac{1}{8T^3}\partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\beta X\partial_\gamma X\cdot\partial_\delta X\partial^\gamma Y\cdot\partial^\delta Y\} \quad . \qquad (4.23)
\end{aligned}
$$

Note that in terms such as $m_F\theta^1\tilde{\gamma}_F\theta^2$ there is an implicit sum over the massive fermions $F=1,\cdots,N_F$.

4.5 Feynman rules

The action (4.23) above yields the following Feynman rules in Minkowski space, where each vertex is accompanied by a momentum delta function and each momentum integral (which we perform in Euclidean space) should be multiplied by i due to Wick rotation.

4.5.1 Propagators

For each fermion with mass m_F the propagator is

$$
\begin{aligned}
G_{ab}(k) &= <\theta^a(k)\theta^b(-k)> \\
&= \frac{4}{p^2+m_F^2}\left[\begin{pmatrix} -ip_- & \frac{m_F}{2}\tilde{\gamma}_F \\ -\frac{m_F}{2}\tilde{\gamma}_F^T & -ip_+ \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1+\gamma^c & \\ & 1\pm\gamma^c \end{pmatrix}\right]_{ab}\times\tilde{\gamma}_F^T\tilde{\gamma}_F \ , \qquad (4.24)
\end{aligned}
$$

where the indices $a,b=1,2$ indicate the entry for the 2×2 matrix. Notice the two projection operators in the fermion propagator. The first is to project the fermion onto the proper 8 dimensional chirality, according to the kappa symmetry fixing (upper sign for

type IIA backgrounds and lower sign for type IIB). The second projector projects onto one d.o.f only, with a specific mass m_F.

For a scalar with mass m_B the propagator is

$$< Y_a(k)Y_b(-k) > = \frac{-i}{k^2+m_B^2}\delta_{ab} \ . \tag{4.25}$$

4.5.2 Interactions

In our conventions, we put into the vertices we write below the sum over permutations of X fields but not of other fields. We will consistently take this into account in the symmetry factors of the loop diagrams. The vertices including only scalar fields are:

$$\begin{aligned}&< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4) >= \delta_{ij}\delta_{kl}\{-\frac{i}{T}k_1\cdot k_2 k_3\cdot k_4+\frac{i}{T}k_1\cdot k_3 k_2\cdot k_4+\frac{i}{T}k_1\cdot k_4 k_2\cdot k_3\}\\&+\delta_{il}\delta_{jk}(k_1\leftrightarrow k_3)+\delta_{ik}\delta_{jl}(k_1\leftrightarrow k_4),\end{aligned} \tag{4.26}$$

$$\begin{aligned}&< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)X_m(k_5)X_n(k_6) >= \delta_{ij}\delta_{kl}\delta_{mn}\{-\frac{3i}{T^2}k_1\cdot k_2 k_3\cdot k_4 k_5\cdot k_6\\&+\frac{i}{T^2}[k_1\cdot k_2(k_3\cdot k_5 k_4\cdot k_6+k_4\cdot k_5 k_3\cdot k_6)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\}\\&+\delta_{ij}\delta_{km}\delta_{ln}(k_3\leftrightarrow k_6)+\delta_{ij}\delta_{kn}\delta_{lm}(k_3\leftrightarrow k_5)+\delta_{kl}\delta_{im}\delta_{jn}(k_1\leftrightarrow k_6)+\delta_{kl}\delta_{in}\delta_{jm}(k_1\leftrightarrow k_5)\\&+\text{ more permutations on }[(i;1),(j;2),(k;3),(l;4),(m;5),(n;6)],\end{aligned} \tag{4.27}$$

$$\begin{aligned}&< X_i(k_1)X_j(k_2)Y_a(k_3)Y_b(k_4) >= \delta_{ij}\delta_{ab}\frac{i}{2T}\{(-k_1\cdot k_2 k_3\cdot k_4+k_1\cdot k_3 k_2\cdot k_4+k_1\cdot k_4 k_2\cdot k_3)\\&+m_B^2 k_1\cdot k_2\},\end{aligned} \tag{4.28}$$

$$\begin{aligned}&< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)Y_a(k_5)Y_b(k_6) >= \delta_{ab}\delta_{ij}\delta_{kl}\frac{i}{2T^2}\{k_1\cdot k_2(k_3\cdot k_5 k_4\cdot k_6+k_3\cdot k_6 k_4\cdot k_5)\\&+k_3\cdot k_4(k_1\cdot k_5 k_2\cdot k_6+k_1\cdot k_6 k_2\cdot k_5)-3k_1\cdot k_2 k_3\cdot k_4 k_5\cdot k_6+k_1\cdot k_3 k_2\cdot k_4 k_5\cdot k_6\\&+k_1\cdot k_4 k_2\cdot k_3 k_5\cdot k_6+m_B^2(-k_1\cdot k_2 k_3\cdot k_4+k_1\cdot k_3 k_2\cdot k_4+k_1\cdot k_4 k_2\cdot k_3)\}\\&+\delta_{ab}\delta_{il}\delta_{jk}(k_1\leftrightarrow k_3)+\delta_{ab}\delta_{ik}\delta_{jl}(k_1\leftrightarrow k_4),\end{aligned} \tag{4.29}$$

$$\begin{aligned}&< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)X_m(k_5)X_n(k_6)Y_a(k_7)Y_b(k_8) >= \delta_{ij}\delta_{kl}\delta_{mn}\delta_{ab}\\&\times\frac{i}{2T^3}\{k_1\cdot k_2 k_3\cdot k_4 k_5\cdot k_6(-9k_7\cdot k_8-3m_B^2)\\&+[k_3\cdot k_4 k_5\cdot k_6(k_1\cdot k_7 k_2\cdot k_8+k_2\cdot k_7 k_1\cdot k_8)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\\&+(k_7\cdot k_8+m_B^2)[k_1\cdot k_2(k_3\cdot k_5 k_4\cdot k_6+k_3\cdot k_6 k_4\cdot k_5)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\\&+[(k_5\cdot k_3 k_6\cdot k_4+k_5\cdot k_4 k_6\cdot k_3)(k_1\cdot k_7 k_2\cdot k_8+k_1\cdot k_8 k_2\cdot k_7)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\}\\&+\text{ permutations on }[(i;1),(j;2),(k;3),(l;4),(m;5),(n;6)] \quad .\end{aligned} \tag{4.30}$$

The vertices involving fermions are, using $k_i\times k_j\equiv\epsilon^{\alpha\beta}k_{i\alpha}k_{j\beta}$,

$$\begin{aligned}&< X_i(k_1)X_j(k_2)\theta^1(k_3)\theta^1(k_4) >= \delta_{ij}\frac{i}{8T}(-k_1\cdot k_2(k_{4+}-k_{3+})+k_2\cdot(k_4-k_3)k_{1+}\\&+k_1\cdot(k_4-k_3)k_{2+}),\end{aligned} \tag{4.31}$$

$$< X_i(k_1)X_j(k_2)\theta^2(k_3)\theta^2(k_4) >= \delta_{ij}\frac{i}{8T}(-k_1\cdot k_2(k_{4-}-k_{3-})+k_2\cdot(k_4-k_3)k_{1-}$$
$$+k_1\cdot(k_4-k_3)k_{2-}), \tag{4.32}$$

$$< X_i(k_1)X_j(k_2)\theta^1(k_3)\theta^2(k_4) >= -\frac{m_F\tilde{\gamma}_F}{8T}k_1\cdot k_2-\frac{m_F[\gamma_i,\gamma_j]\tilde{\gamma}_F}{16T}k_1\times k_2, \tag{4.33}$$

$$< X_i(k_1)X_j(k_2)\theta^2(k_3)\theta^1(k_4) >= \frac{m_F\tilde{\gamma}_F^T}{8T}k_1\cdot k_2-\frac{m_F[\gamma_i,\gamma_j]\tilde{\gamma}_F^T}{16T}k_1\times k_2, \tag{4.34}$$

$$< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)\theta^1(k_5)\theta^1(k_6) >= \delta_{ij}\delta_{kl}\frac{i}{8T^2}\{k_1\cdot k_2k_3\cdot(k_6-k_5)k_{4+}$$
$$+k_1\cdot k_2k_4\cdot(k_6-k_5)k_{3+}+k_3\cdot k_4k_1\cdot(k_6-k_5)k_{2+}+k_3\cdot k_4k_2\cdot(k_6-k_5)k_{1+}$$
$$-3k_1\cdot k_2k_3\cdot k_4(k_{6+}-k_{5+})+k_1\cdot k_3k_2\cdot k_4(k_{6+}-k_{5+}),+k_1\cdot k_4k_2\cdot k_3(k_{6+}-k_{5+})\}$$
$$+\delta_{il}\delta_{jk}(k_1\leftrightarrow k_3)+\delta_{ik}\delta_{jl}(k_1\leftrightarrow k_4), \tag{4.35}$$

$$< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)\theta^2(k_5)\theta^2(k_6) >= \delta_{ij}\delta_{kl}\frac{i}{8T^2}\{k_1\cdot k_2k_3\cdot(k_6-k_5)k_{4-}$$
$$+k_1\cdot k_2k_4\cdot(k_6-k_5)k_{3-}+k_3\cdot k_4k_1\cdot(k_6-k_5)k_{2-}+k_3\cdot k_4k_2\cdot(k_6-k_5)k_{1-}$$
$$-3k_1\cdot k_2k_3\cdot k_4(k_{6-}-k_{5-})+k_1\cdot k_3k_2\cdot k_4(k_{6-}-k_{5-})+k_1\cdot k_4k_2\cdot k_3(k_{6-}-k_{5-})\}$$
$$+\delta_{il}\delta_{jk}(k_1\leftrightarrow k_3)+\delta_{ik}\delta_{jl}(k_1\leftrightarrow k_4), \tag{4.36}$$

$$< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)\theta^1(k_5)\theta^2(k_6) >= \delta_{ij}\delta_{kl}\frac{m_F\tilde{\gamma}_F}{8T^2}(k_1\cdot k_2k_3\cdot k_4$$
$$-k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)+\delta_{il}\delta_{jk}(k_1\leftrightarrow k_3)+\delta_{ik}\delta_{jl}(k_1\leftrightarrow k_4), \tag{4.37}$$

$$< X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)\theta^2(k_5)\theta^1(k_6) >= -\delta_{ij}\delta_{kl}\frac{m_F\tilde{\gamma}_F^T}{8T^2}(k_1\cdot k_2k_3\cdot k_4$$
$$-k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)+\delta_{il}\delta_{jk}(k_1\leftrightarrow k_3)+\delta_{ik}\delta_{jl}(k_1\leftrightarrow k_4)\quad . \tag{4.38}$$

5. Examples

In this section we review some known confining backgrounds which have a dual gauge theory interpretation. All of these examples are special cases of the general background which we analyze in this paper. We provide a very short description for each background, followed by a derivation of the physical parameters of the background (scalar and fermion masses). In all of these examples, one can see that the sum rule (2.16) holds.

5.1 Witten background for $D = 3$

A confining theory related to pure $SU(N)$ Yang-Mills (YM) theory in 3 dimensions was proposed in [18]. The approach taken there was to start with the $\mathrm{AdS}_5/\mathrm{CFT}_4$ duality and to compactify the conformal theory on a circle with radius R_0, taking anti-periodic boundary conditions for the fermions. This breaks supersymmetry explicitly, and the fermions all

have a mass proportional to $1/R_0$. The 3 dimensional coupling is $g_3 = g_4^2 N/R_0$. The pure YM theory is obtained in the limit $g_4^2 N \to 0$, $R_0 \to 0$, g_3 fixed. However, with present knowledge, we can only analyze the gravity side at small curvature, which implies $g_4^2 N >> 1$. Therefore the theory we analyze is dual to a strongly coupled 3 dimensional theory (which becomes four dimensional at the scale $1/R_0$; this turns out to also be the scale of the mass gap at strong coupling).

On the gravity side, the theory is a type IIB superstring theory. The background is given [18] by a metric and a five-form,

$$\frac{ds^2}{R^2} = (u^2 - \frac{u_0^4}{u^2})d\tau^2 + (u^2 - \frac{u_0^4}{u^2})^{-1}du^2 + u^2\sum_{i=0}^{2} dX_i^2 + d\Omega_5^2,$$
$$F_5 = 16\pi N\alpha'^2\omega_5, \quad R^2 = \sqrt{4\pi g_s N}\alpha', \quad \int_{S^5}\omega_5 = \pi^3, \quad e^\phi = g_s, \tag{5.1}$$

where u_0 is related to the periodicity of the circle coordinate τ, and ω_5 is the volume form on a 5-sphere with unit radius. The τ circle shrinks smoothly at $u = u_0$, ending the space. In this background, as we will show in the next subsections, there are six massless scalars, two massive scalars with mass $\frac{m_B^2}{T} = \sqrt{\frac{16\pi}{g_s N}}$ and eight massive fermions with mass $\frac{m_F^2}{T} = \sqrt{\frac{\pi}{g_s N}}$.

5.1.1 Scalar masses

We take the limit $u \to u_0$:

$$u = u_0(1 + \frac{2\pi\alpha'}{R^2}\rho^2 + O(\rho^4)),$$
$$\frac{ds^2}{2\pi\alpha'} = (1 + \frac{4\pi\alpha'}{R^2}\rho^2)\sum_{i=0}^{2} dX_i^2 + \rho^2 d\tau^2 + d\rho^2 + \frac{R^2}{2\pi\alpha'}d\Omega_5^2, \tag{5.2}$$

where we rescaled X and τ. Comparing the last equation in (5.2) with (4.2) we find two massive scalars of mass $\frac{m_B^2}{T} = \frac{8\pi\alpha'}{R^2} = \sqrt{\frac{16\pi}{g_s N}}$, which are the two radial directions (ρ and τ), and six massless transverse scalars coming from the X and S^5 directions.

5.1.2 Fermion masses

The 5-form in this background couples to the fermions, as described in [34, 35], so that the covariant derivative on the worldsheet is:

$$\begin{aligned}
D_\alpha^{IJ} &= \partial_\alpha\delta^{IJ} + \frac{e^\phi}{16\cdot 5!}F_{abcde}\Gamma^{abcde}\partial_\alpha Z^\mu\Gamma_\mu Q_5^{IJ} = \partial_\alpha\delta^{IJ} + \sqrt{2\pi\alpha'}g_s\frac{2\pi N\alpha'^2}{R^5}\rho\gamma^{56789}\partial_\alpha Z\cdot\Gamma Q_5^{IJ} \\
&= \partial_\alpha\delta^{IJ} + \frac{1}{2}\left(\frac{\pi}{g_s N}\right)^{\frac{1}{4}}\rho\gamma^{56789}\partial_\alpha Z\cdot\Gamma Q_5^{IJ} \\
&\equiv \partial_\alpha\delta^{IJ} + \tilde{\Gamma}_5\partial_\alpha Z\cdot\Gamma Q_5^{IJ} = \partial_\alpha\delta^{IJ} + \sum_{F=1}^{8}\frac{m_F}{2\sqrt{T}}\rho\tilde{\gamma}_F\partial_\alpha Z\cdot\Gamma Q_5^{IJ},
\end{aligned} \tag{5.3}$$

where in $\partial_\alpha Z\cdot\Gamma$ we contract with δ_{ab}. Notice the factor 2 in the second equality coming from the fact that we included the dual 5-form. In this notation $\tilde{\Gamma}_5 = \frac{1}{2}(\frac{\pi}{g_s N})^{\frac{1}{4}}\rho\gamma^{56789}$.

There are 8 massive fermions with mass $\frac{m_F^2}{T} = \sqrt{\frac{\pi}{g_s N}}$. We can take $\tilde{\gamma}_F$ to be γ^{56789} times any basis of projection operators commuting with γ^{56789}.

5.2 Witten background for $D = 4$

A theory related to a pure YM theory in 4 dimensions can be achieved by methods similar to those in the previous example, starting from D4-branes [18]. The string theory is a type IIA theory, with a background including the metric, a 4-form on the sphere, and a dilaton which diverges at infinity. Again, there is a circle which vanishes at a finite radial coordinate. The background is given by [18]

$$\begin{aligned} \frac{ds^2}{\alpha'} &= \frac{2\pi\lambda}{3u_0} u \left(4u^2 \sum_{i=0}^{3} dx_i^2 + \frac{4}{9u_0^2} u^2 (1 - \frac{u_0^6}{u^6}) d\tau^2 + 4 \frac{du^2}{u^2(1 - \frac{u_0^6}{u^6})} + d\Omega_4^2 \right), \\ F_4 &= 3\pi N \alpha'^{\frac{3}{2}} \omega_4, \quad R^2 = \frac{2\pi\lambda}{3}\alpha', \quad \int_{S_4} \omega_4 = \frac{8\pi^2}{3}, \quad e^{2\phi} = \frac{8\pi\lambda^3 u^3}{27 u_0^3 N^2} \quad , \end{aligned} \tag{5.4}$$

where λ is related to the four dimensional 't Hooft coupling, and ω_4 is the volume form on the unit 4-sphere. In this background we find there are six massless scalars, two massive scalars with mass $\frac{m_B^2}{T} = \frac{27}{4\lambda}$ and eight massive fermions with mass $\frac{m_F^2}{T} = \frac{27}{16\lambda}$.

5.2.1 Scalar masses

We take the limit $u \to u_0$, and obtain (after rescalings)

$$\begin{aligned} u &= u_0(1 + \frac{9}{8\lambda}\rho^2 + O(\rho^4)), \\ \frac{ds^2}{2\pi\alpha'} &= (1 + \frac{27}{8\lambda}\rho^2) \sum_{i=0}^{3} dx_i^2 + \rho^2 d\tau^2 + d\rho^2 + \frac{R^2}{2\pi\alpha'} d\Omega_4^2 \quad . \end{aligned} \tag{5.5}$$

Comparing to (4.2) we find six transverse massless scalars and two massive scalars with mass $\frac{m_B^2}{T} = \frac{27}{4\lambda}$.

5.2.2 Fermion masses

The 4-form in this background couples to the fermions, as described in [34, 35], so that the covariant derivative on the worldsheet is

$$\begin{aligned} D_\alpha &= \partial_\alpha + \frac{1}{8 \cdot 4!} e^\phi F_{abcd} \Gamma^{abcd} \partial_\alpha Z^\mu \Gamma_\mu = \partial_\alpha + \frac{1}{8}\sqrt{2\pi\alpha'} \sqrt{\frac{8\pi\lambda^3}{27N^2}} \frac{27N}{4\pi\lambda^2\sqrt{\alpha'}} \gamma^{6789} \partial_\alpha Z \cdot \Gamma \\ &= \partial_\alpha + \frac{1}{2}\sqrt{\frac{27}{16\lambda}} \gamma^{6789} \partial_\alpha Z \cdot \Gamma \equiv \partial_\alpha + \sum_{F=1}^{8} \frac{m_F}{2\sqrt{T}} \tilde{\gamma}_F \partial_\alpha Z \cdot \Gamma \quad . \end{aligned} \tag{5.6}$$

Here we used $\tilde{\Gamma}_4 = \frac{1}{2}\sqrt{\frac{27}{16\lambda}} \gamma^{6789}$. We find 8 massive fermions, each with mass $\frac{m_F^2}{T} = \frac{27}{16\lambda}$.

5.3 The Maldacena-Nuñez background

It was proposed in [19] that the gravity solution found in [25] is associated with the theory of N D5-branes on a 2-sphere, which in a specific limit becomes the four dimensional $\mathcal{N}=1$ supersymmetric YM theory (SYM). The UV theory is 6 dimensional and maximally supersymmetric. The spin structure on the sphere is taken such that only 4 supersymmetries remain, and in the limit of small 't Hooft coupling the IR theory is the 4 dimensional $\mathcal{N}=1$ SYM. In the weakly curved limit (large 't Hooft coupling) there is no separation between the SYM theory and the six dimensional modes. The background consists of a metric, a R-R 3-form, and a dilaton:

$$\begin{aligned}
\frac{ds^2}{\alpha'} &= e^{\phi_D} N[\sum_{i=0}^{3} dx_i^2 + d\rho^2 + e^{2g(\rho)} d\Omega_2^2 + \frac{1}{4}\sum_a (\omega^a - A^a)^2], \\
e^{2\phi_D} &= e^{2\phi_{D,0}} \frac{\sinh(2\rho)}{2e^{g(\rho)}} = g_s^2 (1 + \frac{8}{9}\rho^2 + O(\rho^4)), \\
F_3 &= N[-\frac{1}{4}(\omega^1 - A^1)\wedge(\omega^2 - A^2)\wedge(\omega^3 - A^3) + \frac{1}{4}\sum_a F\wedge(\omega^a - A^a)],
\end{aligned} \tag{5.7}$$

where (see [19] for details)

$$\begin{aligned}
a(\rho) &= \frac{2\rho}{\sinh(2\rho)} = 1 - \frac{2}{3}\rho^2 + O(\rho^4), \\
e^{2g} &= \rho\coth(2\rho) - \frac{\rho^2}{\sinh^2(2\rho)} - \frac{1}{4} = \rho^2 + O(\rho^4), \\
A &= \frac{1}{2}[\sigma^1 a(\rho) d\theta + \sigma^2 a(\rho)\sin(\theta) d\phi + \sigma^3 \cos(\theta) d\phi], \quad F = dA + A\wedge A, \\
\omega^1 + i\omega^2 &= e^{-i\psi}(d\tilde{\theta} + i\sin(\tilde{\theta}) d\Phi), \quad \omega^3 = d\psi + \cos(\tilde{\theta}) d\phi \quad .
\end{aligned} \tag{5.8}$$

There is also a limit of this background where the string coupling becomes strong so one needs to use an S-duality transformation, after which only NS-NS fields are turned on. In this limit the confining string is a D-string, and it belongs in the class of backgrounds discussed in section 2.4; we will not discuss it further here.

In this background there are 3 massive scalars with mass $\frac{m_B^2}{T} = \frac{16\pi}{9g_s N}$ and 5 massless transverse scalars. There are 6 massive fermions with mass $\frac{m_F^2}{T} = \frac{8\pi}{9g_s N}$ and 2 massless fermions, which are Goldstinos for the supersymmetries broken by the string.

5.3.1 Scalar masses

Carefully taking the IR limit $\rho \to 0$, and using 3 Cartesian coordinates Y_B instead of ρ and Ω_2, the metric is:

$$\frac{ds^2}{2\pi\alpha'} = \left(1 + \frac{8\pi Y^2}{9g_s N}\right)\sum_{i=0}^{3} dX_i^2 + \sum_{B=4}^{6} dY_B^2 + \frac{1}{8\pi}\sum_{a=7}^{9}(\omega^{a-6} - A^{a-6})^2 \quad . \tag{5.9}$$

Comparing to (4.2) we find 5 massless scalars and 3 massive scalars with mass $\frac{m_B^2}{T} = \frac{16\pi}{9g_s N}$.

5.3.2 Fermion masses

The covariant derivative contains the following term :

$$\begin{aligned}
\frac{\sqrt{2\pi\alpha'}}{8\cdot 3!}e^{\phi}F_{\mu\nu\rho}e^{\mu}_{a}e^{\nu}_{b}e^{\rho}_{c}\Gamma^{abc}\Gamma\cdot\partial_{\alpha}Z &= -\frac{1}{2}\sqrt{\frac{\pi}{18g_sN}}\rho(\gamma^{457}+\gamma^{468}+\gamma^{569}+3\gamma^{789})\Gamma\cdot\partial_{\alpha}Z \\
&= -\frac{1}{2}\sqrt{\frac{8\pi}{9g_sN}}\rho\gamma^{789}(1-P_{I+}P_{II+})\Gamma\cdot\partial_{\alpha}Z, \\
P_{I\pm}\equiv\frac{1}{2}(1\pm\gamma^{5678}),\quad P_{II\pm}\equiv\frac{1}{2}(1\pm\gamma^{4589}),&\quad P_{III\pm}\equiv\frac{1}{2}(1\pm\gamma^{4679}) \quad . \qquad (5.10)
\end{aligned}$$

The indices on the gamma matrices γ^{ijk} are flat space indices, and their numbers correspond to the directions in the metric (5.9). We defined projection operators $P_{I\pm}$, $P_{II\pm}$ and $P_{III\pm}$, each with half zero eigenvalues, so that each product of three of them projects onto one physical d.o.f. The three projectors commute, so we can block diagonalize them simultaneously and in this basis our 8 fermions split into 8 sectors, according to the projectors eigenvalues $\{P_{I+}=(0,1),\ P_{II+}=(0,1),\ P_{III+}=(0,1)\}$. There are two sectors, $\{1,1,0\}$ and $\{1,1,1\}$, for which the mass matrix vanishes. Thus there are 2 massless fermions, while the other 6 fermions all have the same mass, $\frac{m_F^2}{T}=\frac{8\pi}{9g_sN}$.

5.4 Klebanov-Strassler background

The Klebanov-Strassler background is obtained by considering a collection of N regular and M fractional D3-branes in the geometry of a deformed conifold [20]. The gravity solution includes a $\mathbf{R}^4$ part with a warp factor, and a six dimensional conifold (including the radial direction). There are R-R forms F_3 and F_5, and an NS-NS 2-form B. We refer to [39] for the exact background and write here only the expansion near the minimal radial coordinate. We find that there are 5 massless scalars, 3 massive scalars with mass $\frac{m_B^2}{T}=\frac{4\pi}{3a_0^{3/2}g_sM}$ (where $a_0\approx 0.718$ is computed in [39]), 2 massless fermions (corresponding to the $\mathcal{N}=1$ Goldstinos) and 6 massive fermions with mass $\frac{m_F^2}{T}=\frac{2\pi}{3a_0^{3/2}g_sM}$.

5.4.1 Scalar masses

The metric near $\rho=0$ is

$$\begin{aligned}
\frac{ds^2}{2\pi\alpha'} &= \left(1+\frac{6^{1/3}2\pi a_1}{a_0^{3/2}g_sM}\rho^2\right)\sum_{i=0}^{3}dX_i^2+d\rho^2+\frac{\rho^2}{2}((g^1)^2+(g^2)^2)+\frac{\sqrt{a_0}g_sM\alpha'}{6^{1/3}}((g^3)^2+(g^4)^2) \\
&\quad +\frac{\sqrt{a_0}g_sM\alpha'}{4\cdot 6^{1/3}}(g^5)^2 \quad . \qquad (5.11)
\end{aligned}$$

Here ρ is the radial direction, and g^1 and g^2 are the tangent directions on a 2-sphere which shrinks to zero at $\rho=0$. g^3,g^4,g^5 are other directions on the sphere which are massless. Comparing to (4.2) we find 5 massless scalars and 3 massive scalars with mass $\frac{m_B^2}{T}=\frac{6^{1/3}4\pi a_1}{a_0^{3/2}g_sM}=\frac{4\pi}{3a_0^{3/2}g_sM}$. We used the value $a_1=\frac{6^{2/3}}{18}$ computed in [39].

5.4.2 Fermion masses

Out of the 4 background fields, there are two which do not vanish at the minimal radial coordinate, F_3 and H_3. As we stated in section 4, if H_3 is not polarized along the worldsheet then it does not contribute to the fermion mass terms, which is the case here. Therefore, the only contribution comes from the R-R 3-form, whose value at the minimal radial coordinate is

$$\begin{aligned} F_{\mu\nu\rho}e_a^\mu e_b^\nu e_c^\rho \Gamma^{abc} &= \frac{4}{\sqrt{3}a_0^{3/4}M^{1/2}\alpha'^{1/2}g_s^{3/2}}\rho(3\gamma^{345}+\gamma^{125}+\gamma^{\rho 13}+\gamma^{\rho 24}) \\ &= \frac{4}{\sqrt{3}a_0^{3/4}M^{1/2}\alpha'^{1/2}g_s^{3/2}}\rho\gamma^{345}(1-P_{I+}P_{II+}) \quad . \end{aligned} \tag{5.12}$$

The covariant derivative is therefore

$$\begin{aligned} &\frac{\sqrt{2\pi\alpha'}}{8\cdot 3!}e^\phi F_{\mu\nu\rho}e_a^\mu e_b^\nu e_c^\rho \Gamma^{abc}\Gamma\cdot\partial_\alpha Z = \frac{1}{2}\sqrt{\frac{2\pi}{3a_0^{3/2}g_s M}}\rho\gamma^{345}(1-P_{I+}P_{II+})\Gamma\cdot\partial_\alpha Z \quad , \\ &P_{I\pm}\equiv\frac{1}{2}(1\pm\gamma^{1234}), \quad P_{II\pm}\equiv\frac{1}{2}(1\pm\gamma^{\rho 145}), \quad P_{III\pm}\equiv\frac{1}{2}(1\pm\gamma^{\rho 235}) \quad . \end{aligned} \tag{5.13}$$

The indices on the gamma matrices γ^{ijk} are flat space indices and correspond to the directions in the metric (5.11), where ρ is the radial direction and the indices $1,\cdots,5$ correspond to the $g^1,\cdots,g^5$ directions. The projection operators were defined similarly to those in the Maldacena-Nuñez example. Thus, we find 2 massless fermions, while the other 6 fermions all have the same mass, $\frac{m_F^2}{T}=\frac{2\pi}{3a_0^{3/2}g_s M}$.

6. The effective action from correlation functions

In this section we compute in confining holographic backgrounds of the form discussed above, the low-energy effective worldsheet theory for the X fields, which are the coordinates where the gauge theory lives. The effective action will contain corrections to the classical interactions. We expect the corrections to be a series in powers of $\frac{m^2}{T}$ and $\frac{\partial^2}{T}$, since the loop expansion parameter is $\frac{1}{T}$ [7] (T multiplies the whole action in some normalization of the fields). We can write the effective action as a Nambu-Goto action plus corrections :

$$S=-\int d^2\sigma\{\sqrt{\det(-T'\delta_{\alpha,0}\delta_{\beta,0}+T'\delta_{\alpha,1}\delta_{\beta,1}+\partial_\alpha\tilde{X}\cdot\partial_\beta\tilde{X})}-\mathcal{F}(\tilde{X})\}, \tag{6.1}$$

where we allow some renormalization of the fields, $\tilde{X}=X(1+O(m^2/T))$ and the tension, $T'=T+O(m^2/T)$. In order to evaluate the effective action we compute correlation functions in both the original and the effective theory. By comparing the two, we determine the coefficients of the operators appearing in $\mathcal{F}(\tilde{X})$, whose general form was discussed in section 2. Since we are interested only in corrections depending on m^2, we do not compute the contribution of massless fields to the effective action.

[7] We will see below that this is not precisely correct due to logarithmic divergences.

Our original action (4.23) is non-renormalizable, however all the results we find are finite, up to possible quadratic divergences which we do not calculate as they have contributions from the massless modes. Presumably this is because the theory is really finite in a different gauge (conformal gauge). We use a sharp cutoff regularization with cutoff Λ. The logarithmic convergence of our calculation is important. Such divergences would appear as $\frac{m^2}{T}\log[m^2/\Lambda^2]$, which implies they vanish in flat space, and a priori one may need to add counter-terms in order to have a finite effective theory, and the renormalization procedure will ruin our predictability regarding $\mathcal{F}(\tilde{X})$. Notice that even if there are quadratic divergences, at one loop order they appear as $\frac{\Lambda^2}{T}$ with no powers of m^2, and thus renormalizing these divergences has no finite effect on our result.

We find that the effective action at the six-derivative order actually shows no deviation from Nambu-Goto, namely $\mathcal{F}(\tilde{X}) = 0$ to this order.

6.1 The tension

The tension T' is the constant term in (6.1), which can be determined from the term linear in L in the ground state energy of a closed string of length L. The leading order ground state energy of a string of length L is $E = TL$, where T is the classical tension of the string. There are corrections to the energy coming from quantum fluctuations of the worldsheet fields. At one-loop order this is given by a summation of all (on-shell) modes in the worldsheet theory [22],

$$\begin{aligned} E &= TL + \frac{\pi}{4L}\sum_{n=-\infty}^{\infty}\{(N_B^0 - N_F^0)|n| + \sum_B \sqrt{n^2 + \frac{m_B^2 L^2}{\pi^2}} - \sum_F \sqrt{n^2 + \frac{m_F^2 L^2}{\pi^2}}\} \\ &= TL + \frac{L}{8\pi}\{\sum_F m_F^2 \log(m_F^2) - \sum_B m_B^2 \log(m_B^2)\} + O(L^0). \end{aligned} \tag{6.2}$$

Here we approximated the sum as an integral, which is correct up to $\frac{1}{L}$ corrections. Interpreting terms linear in L as corrections to the string tension, we find $T' = T + \Delta T$ with

$$\Delta T = \frac{1}{8\pi}\{\sum_F m_F^2 \log(m_F^2) - \sum_B m_B^2 \log(m_B^2)\}. \tag{6.3}$$

Note that the logs appearing in (6.2) are really $\log(\frac{m^2}{\Lambda^2})$, but the cutoff dependence cancels out exactly when $\sum_F m_F^2 = \sum_B m_B^2$, and we will assume this from here on.

6.2 2-point function

In this subsection we compute $\Pi(k) \equiv \langle X^i(k) X^j(-k)\rangle$ to see if we need to perform a wave-function renormalization of X in order to obtain the quadratic term in (6.1).

6.2.1 Fermion diagrams

There are two fermionic diagrams contributing to the propagator $\Pi(k)$. One should be careful about the Wick contractions, paying attention to various minus signs. For example, a 4-point diagram involving two vertices of the form $XX\theta^1\tilde{\gamma}\theta^2$ is evaluated with the

following fermionic trace:

$$< X^i X^j \theta_a^1 \tilde{\gamma}^{ab} \theta_b^2 X^k X^l \theta_c^1 \tilde{\gamma}^{cd} \theta_d^2 > = < X^i X^j X^k X^l \theta_a^1 (k-p) \tilde{\gamma}^{ab} \theta_b^2 (p) \theta_c^1 (-\tilde{p}) \tilde{\gamma}^{cd} \theta_d^2 (\tilde{p}-k) >$$
$$\propto \mathrm{tr}[-G_{21}(p-k)\tilde{\gamma} G_{21}(p)\tilde{\gamma} + G_{11}(p-k)\tilde{\gamma} G_{22}(p)\tilde{\gamma}^T] \quad (6.4)$$

where $G_{ab}(p)$ are the entries in the fermion propagator matrix (4.24).

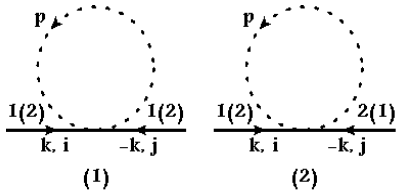

Figure 3: The 2-point fermion diagrams: (1) $\Delta\Pi_1$, (2) $\Delta\Pi_2$. The fermionic propagator is marked by a dashed line. The numbers indicate the propagator indices; e.g. in (2) there are two contributions coming from G_{12} and G_{21}. External solid lines mark the incoming scalars X^i and X^j, with momenta k and $-k$ respectively.

We need to evaluate the diagrams of figure 3, where the indices on the fermion loops refer to the type of fermion (θ^1 or θ^2) in the loop. The first diagram has no contribution,

$$\begin{aligned}
\Delta\Pi_1 &= \sum_F \frac{i}{4T}\delta_{ij} \int \frac{d^2p}{(2\pi)^2} \{(-p_+ k^2 + 2k_+ k\cdot p)(-ip_-)\mathrm{tr}[(1+\gamma^c)\tilde{\gamma}_F^T \tilde{\gamma}_F] \\
&+ (-p_- k^2 + 2k_- k\cdot p)(-ip_+)\mathrm{tr}[(1 \pm \gamma^c)\tilde{\gamma}_F^T \tilde{\gamma}_F]\} \frac{1}{2}\frac{4}{p^2+m_F^2} \\
&= \sum_F \frac{1}{2T}\delta^{ij} \int \frac{d^2p}{(2\pi)^2} (\frac{1}{2}p^2k^2 - (k\cdot p)^2) \frac{1}{p^2+m_F^2} = 0 \quad .
\end{aligned} \tag{6.5}$$

The second fermionic diagram is given by

$$\begin{aligned}
\Delta\Pi_2 &= \sum_F \frac{m_F}{8T}(-k^2)\frac{m_F}{2}\delta^{ij} \int \frac{d^2p}{(2\pi)^2} \frac{4}{p^2+m_F^2}\frac{1}{2}\{-\mathrm{tr}[\tilde{\gamma}_F^T(1\pm\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F\tilde{\gamma}_F] \\
&- \mathrm{tr}[\tilde{\gamma}^F(1+\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}^F\tilde{\gamma}_F^T]\} = \sum_F \frac{m_F^2}{2T}k^2\delta^{ij} \int \frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m_F^2} \\
&= \sum_F \frac{im_F^2}{8\pi T}k^2\delta^{ij}\log[\frac{\Lambda^2}{m_F^2}] \quad .
\end{aligned} \tag{6.6}$$

Here we used $\mathrm{tr}[\tilde{\gamma}_F^T\tilde{\gamma}_F\gamma^c] = 0$ and (4.16). The i in the last line comes from computing the integral in Euclidean space. Because of our $(-,+)$ signature choice, $p^2 = p_E^2$, and the only change is a factor of i from setting $p_0 = ip_E$. Note that we evaluate all other diagrams in the same way.

6.2.2 Scalar diagram

There is a single scalar diagram,

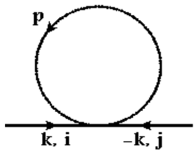

Figure 4: The 2-point scalar diagram $\Delta\Pi_3$. Both massive and massless scalars are marked by solid lines.

$$\begin{aligned}\Delta\Pi_3 &= \sum_B \frac{i}{2T}\delta^{ij}\int\frac{d^2p}{(2\pi)^2}(-m_B^2k^2 - k^2p^2 + 2(k\cdot p)^2)\frac{-i}{p^2+m_B^2} \\ &= \sum_B -\frac{m_B^2k^2}{2T}\delta^{ij}\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m_B^2} = \sum_B -\frac{im_B^2}{8\pi T}k^2\delta^{ij}\log[\frac{\Lambda^2}{m_B^2}] \quad . \end{aligned} \tag{6.7}$$

6.2.3 Conclusion

The sum of all diagrams is,

$$\Delta\Pi = \frac{ik^2}{8\pi T}\{\sum_B m_B^2\log(m_B^2) - \sum_F m_F^2\log(m_F^2)\} = -ik^2\frac{\Delta T}{T} \quad . \tag{6.8}$$

We see that the logarithmic divergence cancels between the fermionic and scalar diagrams and we are left with a finite contribution. We can now obtain the two-point function:

$$\begin{aligned}< X(k)X(-k) > &= -\frac{i}{k^2} + \Delta\Pi(-\frac{i}{k^2})^2 = -\frac{i}{k^2}\left(1 - \frac{i\Delta\Pi}{k^2}\right) \\ &= -\frac{i}{k^2(1+\frac{i\Delta\Pi}{k^2})} = -\frac{i}{k^2(1+\frac{\Delta T}{T})}. \end{aligned} \tag{6.9}$$

This result, together with the tension correction found in section 6.1, are obtained from the following Minkowskian effective action,

$$\begin{aligned}S &= -\int d^2\sigma\{T + \Delta T + \frac{1}{2}\left(1+\frac{\Delta T}{T}\right)\partial_\alpha X\partial^\alpha X\} \qquad (6.10) \\ &= -\int d^2\sigma\{\sqrt{-\det[(T+\Delta T)(-\delta_{\alpha,0}\delta_{\beta,0}+\delta_{\alpha,1}\delta_{\beta,1}) + \partial_\alpha\tilde{X}\cdot\partial_\beta\tilde{X}]} + O(k^4/T^2)\} \quad .\end{aligned}$$

We find that to this order the effective action is the NG action, with an effective tension $T' = T + \Delta T$, and rescaled fields $\tilde{X}^i = (1+\frac{\Delta T}{2T})X^i$ (to leading order in ΔT).

6.3 4-point function

In this subsection we compute the 4-point function $\langle X^i(k_1)X^j(k_2)X^k(k_3)X^l(k_4)\rangle$. We perform the calculation first at leading order in the external momenta, to extract the leading term $O(\frac{k^4}{T^2})$. For further simplicity, we only consider terms proportional to $\delta_{ij}\delta_{kl}$, since the other terms follow by permutations; in most diagrams this means that only vertices where X^i is paired with X^j and X^k is paired with X^l contribute, so we only evaluate these.

6.3.1 Fermion diagrams

We start from diagrams with a fermion loop. We compute in turn the contribution of each diagram to the 4-point function. In some cases we write the expressions for momentum integrals for arbitrary dimension d, to help keep track of numerical factors. At the end of the computation we always set $d = 2$. The first diagram is

$$\begin{aligned} M^4_{F1} &= (-2) \times \delta_{ij}\delta_{kl} \sum_F \int \frac{d^2p}{(2\pi)^2} (\frac{i}{4T})^2 \frac{(-4i)^2}{(p^2+m_F^2)^2} \frac{1}{4} \mathrm{tr}[(1+\gamma^c)^2 \tilde{\gamma}^T_F \tilde{\gamma}^F] \\ &\quad \times \{(-k_1 \cdot k_2 p_+ + k_{1+} k_2 \cdot p + k_{2+} k_1 \cdot p)(p_-)(-k_3 \cdot k_4 p_+ + k_{3+} k_4 \cdot p + k_{4+} k_3 \cdot p)(p_-)\} \\ &= \delta_{ij}\delta_{kl} \sum_F (\frac{-2}{T^2}) \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2+m_F^2)^2} \\ &\quad \times \{(p_+ p_-)^2 k_1 \cdot k_2 k_3 \cdot k_4 + k_{1+} p_- k_{3+} p_- p \cdot k_2 p \cdot k_4 + k_{1+} p_- k_{4+} p_- p \cdot k_2 p \cdot k_3 \\ &\quad + k_{2+} p_- k_{3+} p_- p \cdot k_1 p \cdot k_4 + k_{2+} p_- k_{4+} p_- p \cdot k_1 p \cdot k_3 - k_{1+} p_- p_+ p_- p \cdot k_2 k_3 \cdot k_4 \\ &\quad - k_{2+} p_- p_+ p_- p \cdot k_1 k_3 \cdot k_4 - k_{3+} p_- p_+ p_- p \cdot k_4 k_1 \cdot k_2 - k_{4+} p_- p_+ p_- p \cdot k_3 k_1 \cdot k_2\} \end{aligned} \quad (6.11)$$

There is a symmetry factor of (-2) for the two possible contractions in the loop. This diagram is not Lorentz invariant by itself, but only when we add it to M^4_{F2}; however it is easy to see that each diagram separately vanishes.

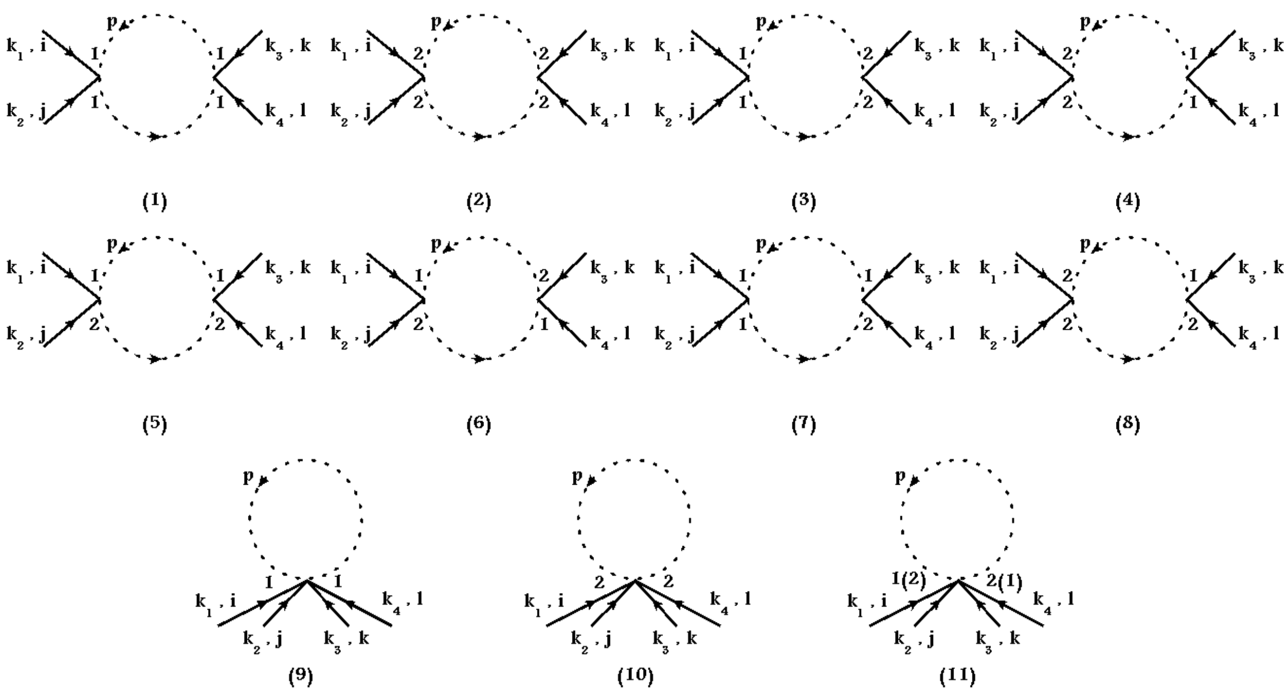

Figure 5: The 4-point fermion diagrams: (1)M^4_{F1}, (2)M^4_{F2}, (3)M^4_{F3}, (4)M^4_{F4}, (5+6)M^4_{F5}+M^4_{F6}, (7)M^4_{F7}, (8)M^4_{F8}, (9)M^4_{F9}, (10)M^4_{F10}, (11)M^4_{F11}.

The following diagram is also not Lorentz invariant by itself,

$$M_{F3}^4 = (-2)\times\sum_F(\frac{i}{4T})^2(\frac{m_F}{2})^2\int\frac{d^2p}{(2\pi)^2}\frac{(-4i)^2}{(p^2+m_F^2)^2}\frac{1}{4}\mathrm{tr}[(1\pm\gamma^c)(\tilde{\gamma}_F)^3(1+\gamma^c)(\tilde{\gamma}_F^T)^3]$$
$$\times\{(-k_1\cdot k_2p_+ + k_{1+}k_2\cdot p + k_{2+}k_1\cdot p)(-k_3\cdot k_4p_- + k_{3-}k_4\cdot p + k_{4-}k_3\cdot p)\}\delta_{ij}\delta_{kl}$$
$$= \delta_{ij}\delta_{kl}\sum_F(-\frac{m_F^2}{2T^2})\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_F^2)^2}\times\{k_1\cdot k_2k_3\cdot k_4p_+p_- + k_{1+}k_{3-}k_2\cdot pk_4\cdot p$$
$$+k_{2+}k_{3-}k_1\cdot pk_4\cdot p + k_{1+}k_{4-}k_2\cdot pk_3\cdot p + k_{2+}k_{4-}k_1\cdot pk_3\cdot p$$
$$-k_1\cdot k_2p_+k_{3-}k_4\cdot p - k_1\cdot k_2p_+k_{4-}k_3\cdot p - k_3\cdot k_4p_-k_{1+}k_2\cdot p$$
$$-k_3\cdot k_4p_-k_{2+}k_1\cdot p\} \quad . \tag{6.12}$$

Here we used the fact that for type IIA (upper sign) $[\tilde{\gamma},\gamma^c]=0$, while for type IIB (lower sign) $\{\tilde{\gamma},\gamma^c\}=0$. The next diagram is :

$$M_{F4}^4 = \delta_{ij}\delta_{kl}\sum_F(-\frac{m_F^2}{2T^2})\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_F^2)^2}$$
$$\times\{k_1\cdot k_2k_3\cdot k_4p_+p_- + k_{1-}k_{3+}k_2\cdot pk_4\cdot p + k_{2-}k_{3+}k_1\cdot pk_4\cdot p + k_{1-}k_{4+}k_2\cdot pk_3\cdot p +$$
$$+k_{2-}k_{4+}k_1\cdot pk_3\cdot p - k_1\cdot k_2p_-k_{3+}k_4\cdot p - k_1\cdot k_2p_-k_{4+}k_3\cdot p -$$
$$k_3\cdot k_4p_+k_{1-}k_2\cdot p - k_3\cdot k_4p_+k_{2-}k_1\cdot p\}. \tag{6.13}$$

Summing the two we obtain the Lorentz-invariant result

$$M_{F3}^4 + M_{F4}^4 = \delta_{ij}\delta_{kl}\sum_F(-\frac{m_F^2}{2T^2})\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_F^2)^2}p^2$$
$$\times\{k_1\cdot k_2k_3\cdot k_4\left(-\frac{1}{2}+\frac{2}{d}\right) + k_1\cdot k_4k_2\cdot k_3(-\frac{1}{d}) + k_1\cdot k_3k_2\cdot k_4(-\frac{1}{d})\}$$
$$= \delta_{ij}\delta_{kl}\sum_F(-\frac{im_F^2}{16\pi T^2})\left(-1+\log[\frac{\Lambda^2}{m_F^2}]\right)$$
$$\times\{k_1\cdot k_2k_3\cdot k_4 - k_1\cdot k_4k_2\cdot k_3 - k_1\cdot k_3k_2\cdot k_4\} \quad . \tag{6.14}$$

In the evaluation of M_{F5}^4 the usual contraction gives a more general index structure,

$$M_{F5}^4 = 4\times(\frac{m_F}{16T})^2\int\frac{d^2p}{(2\pi)^2}\frac{16}{(p^2+m_F^2)^2}k_1\times k_2k_3\times k_4 \tag{6.15}$$
$$\times\frac{1}{4}(-(\frac{m_F}{2})^2\mathrm{tr}[[\gamma_i,\gamma_j]\tilde{\gamma}_F(1\pm\gamma^c)\tilde{\gamma}_F^T[\gamma_k\,\gamma_l]\tilde{\gamma}_F(1\pm\gamma^c)\tilde{\gamma}_F^T]$$
$$+p_+p_-\mathrm{tr}[[\gamma_i,\gamma_j]\tilde{\gamma}_F(1+\gamma^c)[\gamma_k,\gamma_l]\tilde{\gamma}_F^T(1\pm\gamma^c)])$$
$$= \sum_F(-\frac{im_F^2}{16\pi T^2})(k_1\cdot k_4k_2\cdot k_3 - k_1\cdot k_3k_2\cdot k_4)(\delta_{il}\delta_{jk}-\delta_{ik}\delta_{jl})\log[\frac{\Lambda^2}{m_F^2}]$$
$$\Rightarrow \sum_F(-\frac{im_F^2}{16\pi T^2})(2k_1\cdot k_2k_3\cdot k_4 - k_1\cdot k_3k_2\cdot k_4 - k_1\cdot k_4k_2\cdot k_3)\delta_{ij}\delta_{kl}\log[\frac{\Lambda^2}{m_F^2}] \quad .$$

In the last line, we applied the permutations $(l,3)\leftrightarrow(j,2)$ and $(l,3)\leftrightarrow(i,1)$ to obtain the terms proportional to $\delta_{ij}\delta_{kl}$. There is a symmetry factor of (4) due to 4 possible

combinations of our 2 vertices. Each contraction in the loop gives a different contribution as can be seen from the two terms in the parenthesis. We also used

$$\begin{aligned}
&\mathrm{tr}[[\gamma_i,\gamma_j][\gamma_k,\gamma_l]] = 64(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) \quad ,\\
&k_1 \times k_2 k_3 \times k_4 = k_1 \cdot k_4 k_2 \cdot k_3 - k_1 \cdot k_3 k_2 \cdot k_4,\\
&\mathrm{tr}[[\gamma_i,\gamma_j][\gamma_k,\gamma_l]\tilde{\gamma}_F^T\tilde{\gamma}_F] = \frac{1}{8}\mathrm{tr}[[\gamma_i,\gamma_j][\gamma_k,\gamma_l]] \quad .
\end{aligned} \tag{6.16}$$

Next, we have

$$\begin{aligned}
M^4_{F6} =&\ 4\times\delta_{ij}\delta_{kl}\sum_F(\frac{m_F}{8T})^2 k_1\cdot k_2 k_3\cdot k_4\int\frac{d^2p}{(2\pi)^2}\frac{16}{(p^2+m_F^2)^2}\\
&\times\frac{1}{4}\{-p_+p_-\mathrm{tr}[\tilde{\gamma}_F(1\pm\gamma^c)\tilde{\gamma}_F^T(1+\gamma^c)] - \frac{m_F^2}{4}\mathrm{tr}[\tilde{\gamma}_F(1-\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F(1-\gamma^c)\tilde{\gamma}_F^T]\}\\
=&\ \delta_{ij}\delta_{kl}\sum_F\frac{m_F^2}{4T^2}k_1\cdot k_2 k_3\cdot k_4\int\frac{d^2p}{(2\pi)^2}\frac{p^2-m_F^2}{(p^2+m_F^2)^2}\\
=&\ \delta_{ij}\delta_{kl}\sum_F\frac{im_F^2}{16\pi T^2}(-2+\log[\frac{\Lambda^2}{m_F^2}])k_1\cdot k_2 k_3\cdot k_4 \quad .
\end{aligned} \tag{6.17}$$

There is a symmetry factor of 4 for the four possible combinations of two vertices.

The following diagrams vanish after setting $d = 2$,

$$\begin{aligned}
&M^4_{F7} + M^4_{F8} = 8\times\delta_{ij}\delta_{kl}\sum_F(-\frac{m_F}{2})(-\frac{m_F}{8T})(\frac{i}{4T})\int\frac{d^2p}{(2\pi)^2}\frac{(-2i)(2)}{(p^2+m_F^2)^2}\\
&\times\{\mathrm{tr}[\tilde{\gamma}_F^T(1\pm\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F(1+\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F\tilde{\gamma}_F] - \mathrm{tr}[(1+\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F\tilde{\gamma}_F^T(1\pm\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F\tilde{\gamma}_F^T]\}\\
&\times\{(-k_1\cdot k_2p_+ + k_2\cdot pk_{1+} + k_1\cdot pk_{2+})p_-k_3\cdot k_4 + (-k_3\cdot k_4p_+ + k_3\cdot pk_{4+} + k_4\cdot pk_{3+})p_-k_1\cdot k_2\\
&+(-k_1\cdot k_2p_- + k_2\cdot pk_{1-} + k_1\cdot pk_{2-})p_+k_3\cdot k_4 + (-k_3\cdot k_4p_- + k_3\cdot pk_{4-} + k_4\cdot pk_{3-})p_+k_1\cdot k_2\}\\
&\qquad = 8\times\delta_{ij}\delta_{kl}\sum_F(-\frac{m_F^2}{16T^2})\mathrm{tr}[(\pm(\gamma^c)^2-(\gamma^c)^2)\tilde{\gamma}_F^T\tilde{\gamma}_F]\int\frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_F^2)^2}\\
&\qquad \times\frac{1}{2}\{(k_1\cdot k_2p^2 - 2k_2\cdot pk_1\cdot p)k_3\cdot k_4 + (k_3\cdot k_4p^2 - 2k_3\cdot pk_4\cdot p)k_1\cdot k_2\} = 0 \quad .
\end{aligned} \tag{6.18}$$

The following diagrams are single vertex diagrams :

$$\begin{aligned}
M^4_{F9} =&\ \delta_{ij}\delta_{kl}\sum_F\frac{i}{4T^2}\int\frac{d^2p}{(2\pi)^2}\{k_1\cdot k_2k_3\cdot pk_{4+} + k_1\cdot k_2k_4\cdot pk_{3+} + k_3\cdot k_4k_1\cdot pk_{2+} + k_3\cdot k_4k_2\cdot pk_{1+}\\
&- 3k_1\cdot k_2k_3\cdot k_4p_+ + k_1\cdot k_3k_2\cdot k_4p_+ + k_1\cdot k_4k_2\cdot k_3p_+\}\frac{4ip_-}{p^2+m_F^2}\frac{1}{2}\mathrm{tr}[(1+\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F] \quad .
\end{aligned} \tag{6.19}$$

$$\begin{aligned}
M^4_{F10} =&\ \delta_{ij}\delta_{kl}\sum_F\frac{i}{4T^2}\int\frac{d^2p}{(2\pi)^2}\{k_1\cdot k_2k_3\cdot pk_{4-} + k_1\cdot k_2k_4\cdot pk_{3-} + k_3\cdot k_4k_1\cdot pk_{2-} + k_3\cdot k_4k_2\cdot pk_{1-}\\
&- 3k_1\cdot k_2k_3\cdot k_4p_- + k_1\cdot k_3k_2\cdot k_4p_- + k_1\cdot k_4k_2\cdot k_3p_-\}\frac{4ip_+}{p^2+m_F^2}\frac{1}{2}\mathrm{tr}[(1\pm\gamma^c)\tilde{\gamma}_F^T\tilde{\gamma}_F] \quad .
\end{aligned} \tag{6.20}$$

Summing the diagrams M^4_{F9} and M^4_{F10} we obtain

$$M^4_{F9}+M^4_{F10}=\delta_{ij}\delta_{kl}\sum_F(-\frac{1}{8})\frac{4}{T^2}(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)\int\frac{d^2p}{(2\pi)^2}\frac{p^2}{p^2+m_F^2}$$
$$=\delta_{ij}\delta_{kl}\sum_F(-\frac{im_F^2}{8\pi T^2})(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)\left(\frac{\Lambda^2}{m_F^2}-\log[\frac{\Lambda^2}{m_F^2}]\right).\quad(6.21)$$

Finally,

$$M^4_{F11}=2\times\delta_{ij}\delta_{kl}\sum_F\frac{m_F}{8T^2}(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)(2m_F)$$
$$\times\int\frac{d^2p}{(2\pi)^2}\frac{1}{p^2+m_F^2}\frac{1}{2}\mathrm{tr}[\tilde\gamma_F^T\tilde\gamma_F(1+\gamma^c)]$$
$$=\delta_{ij}\delta_{kl}\sum_F\frac{im_F^2}{8\pi T^2}(k_1\cdot k_2k_3\cdot k_4-k_1\cdot k_3k_2\cdot k_4-k_1\cdot k_4k_2\cdot k_3)\log[\frac{\Lambda^2}{m_F^2}].\quad(6.22)$$

There is a symmetry factor of (2) for 2 vertices, $\theta^1\tilde\gamma\theta^2$ and $\theta^2\tilde\gamma\theta^1$.

6.3.2 Scalar diagrams

Next, we compute the diagrams with the scalar fields Y_B running in the loop. There are two diagrams at one-loop order (see figure 6) :

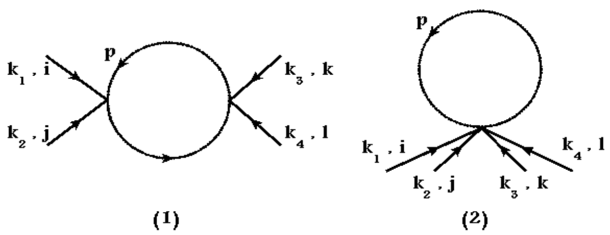

Figure 6: The 4-point scalar diagrams: $(1)M^4_{B1}$, $(2)M^4_{B2}$.

$$M^4{}_{B1}=2\delta_{ij}\delta_{kl}\sum_B\int\frac{d^2p}{(2\pi)^2}\{\frac{m_B^2i}{2T}k_1\cdot k_2+\frac{i}{2T}k_1\cdot k_2p^2-\frac{i}{T}k_1\cdot pk_2\cdot p\}$$
$$\times\{\frac{m_B^2i}{2T}k_3\cdot k_4+\frac{i}{2T}k_3\cdot k_4p^2-\frac{i}{T}k_3\cdot pk_4\cdot p\}\left(\frac{-i}{p^2+m_B^2}\right)^2$$
$$=2\delta_{ij}\delta_{kl}\sum_B\int\frac{d^2p}{(2\pi)^2}\left(\frac{1}{p^2+m_B^2}\right)^2\{k_1\cdot k_2k_3\cdot k_4(\frac{m_B^4}{4T^2}-\frac{1}{8T^2}p^4)$$
$$+\frac{1}{8T^2}(k_1\cdot k_3k_2\cdot k_4+k_1\cdot k_4k_2\cdot k_3)p^4\}$$
$$=\delta_{ij}\delta_{kl}\sum_B\frac{im_B^2}{8\pi T^2}\{k_1\cdot k_2k_3\cdot k_4+\left(\frac{\Lambda^2}{2m_B^2}+\frac{1}{2}-\log[\frac{\Lambda^2}{m_B^2}]\right)$$
$$\times(-k_1\cdot k_2k_3\cdot k_4+k_1\cdot k_3k_2\cdot k_4+k_1\cdot k_4k_2\cdot k_3)\}\quad,\quad(6.23)$$

with a symmetry factor of (2) for 2 possible contractions in the loop, and

$$\begin{aligned}
M^4{}_{B2} &= \delta_{ij}\delta_{kl}\sum_B \frac{i}{2T^2}\int\frac{d^2p}{(2\pi)^2}\{-2k_1\cdot k_2 k_3\cdot p k_4\cdot p - 2k_3\cdot k_4 k_1\cdot p k_2\cdot p \\
&+ p^2(3k_1\cdot k_2 k_3\cdot k_4 - k_1\cdot k_3 k_2\cdot k_4 - k_1\cdot k_4 k_2\cdot k_3) \\
&+ m_B^2(-k_1\cdot k_2 k_3\cdot k_4 + k_1\cdot k_3 k_2\cdot k_4 + k_1\cdot k_4 k_2\cdot k_3)\}\frac{-i}{p^2+m_B^2} \\
&= \delta_{ij}\delta_{kl}\sum_B \frac{1}{2T^2}(k_1\cdot k_2 k_3\cdot k_4 - k_1\cdot k_3 k_2\cdot k_4 - k_1\cdot k_4 k_2\cdot k_3)\int\frac{d^2p}{(2\pi)^2}\{p^2-m_B^2\}\frac{1}{p^2+m_B^2} \\
&= \delta_{ij}\delta_{kl}\sum_B \frac{im_B^2}{4\pi T^2}(k_1\cdot k_2 k_3\cdot k_4 - k_1\cdot k_3 k_2\cdot k_4 - k_1\cdot k_4 k_2\cdot k_3)\left(\frac{\Lambda^2}{2m_B^2} - \log[\frac{\Lambda^2}{m_B^2}]\right). \quad (6.24)
\end{aligned}$$

6.3.3 Conclusion

If we sum our results, ignoring the quadratic divergence, we find the finite result

$$\begin{aligned}
\langle X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)\rangle &= -\delta_{ij}\delta_{kl}\frac{i\Delta T}{T^2}(k_1\cdot k_2 k_3\cdot k_4 - k_1\cdot k_3 k_2\cdot k_4 - k_1\cdot k_4 k_2\cdot k_3) \\
&+ (i,1)\leftrightarrow(k,3) + (i,1)\leftrightarrow(l,4). \quad (6.25)
\end{aligned}$$

Note that the finite contribution proportional to $\frac{1}{16\pi T^2}(\sum_B m_B^2 - \sum_F m_F^2)$ vanished due to our constraint. The quadratic divergence does not vanish in the same manner, and we believe it will vanish once we include loops of other fields (which are independent of m), such as the metric and the kappa gauge-fixing ghosts. This 4-point function, with the addition of the tree-level result (4.26), is generated by the Minkowskian effective action:

$$\begin{aligned}
S_4 &= -(\frac{1}{T}+\frac{\Delta T}{T^2})\int d^2\sigma\{\frac{1}{8}(\partial^\alpha X\cdot\partial_\alpha X)^2 - \frac{1}{4}\partial_\alpha X\cdot\partial_\beta X\partial^\alpha X\cdot\partial^\beta X\} \\
&= -\frac{1}{T'}\int d^2\sigma\{\frac{1}{8}(\partial^\alpha\tilde{X}\cdot\partial_\alpha\tilde{X})^2 - \frac{1}{4}\partial_\alpha\tilde{X}\cdot\partial_\beta\tilde{X}\partial^\alpha\tilde{X}\cdot\partial^\beta\tilde{X}\} \quad (6.26)
\end{aligned}$$

We see that the in terms of the rescaled fields $\tilde{X}$ and tension T' the effective action is precisely the NG action (6.1) expanded to fourth order in derivatives. This is expected from the analysis of section 3, where the effective action was constrained to be the NG action to this order, but in our computation it arises non-trivially. Note that for $D = 3$ the two terms in (6.26) are the same, but our analysis is still valid.

6.4 4-point function: higher derivative corrections

In the previous subsection we calculated the 4-point function to lowest order in the external momenta, so effectively we did the loop computations for zero external momenta. In this subsection we are interested in the corrections at six-derivative order. We will compute the 4-point function exactly as a function of the momenta, and then expand it in powers of k to extract this. To simplify our calculation, we take all the external momenta to be on-shell ($k_i^2 = 0$). This is possible since contributions that are not on-shell will create terms in the effective action that are proportional to the equation of motion, and we know such terms can always be swallowed by field redefinitions, and therefore do not contribute to

the partition function. We use the Mandelstam variables, $s = (k_1 + k_2)^2$, $t = (k_1 + k_3)^2$, $u = (k_1 + k_4)^2$, and introduce the variable $k = k_1 + k_2$ for the incoming momentum in a specific channel. We have the on-shell relation $s+t+u = 0$, and two-dimensional kinematics implies that also $stu = 0$ (we used this above in arguing that the c_5 term in (2.4) is trivial).

Apart from the UV divergences, which we expect to cancel between fermion and scalar diagrams, we expect to find a branch-cut at $s = -4m^2$ for diagrams with a field of mass m running in the loop. This branch cut, indicating that the fields running in the loop become on-shell, is typical for $2 \to 2$ scattering.

Naively, by power counting, at one-loop the six-derivative terms should be independent of m so we are not interested in these terms (since we are only interested in m-dependent contributions). However, a dependence on m can appear through logarithmic divergences, and so we should carefully analyze the diagrams that were quadratically divergent at four-derivative order, and thus, may be logarithmically divergent at six-derivative order. In our case these are the diagrams $M^4_{F1,F2}$ and M^4_{B1}. There are other, single vertex diagrams, which are also quadratically divergent, but in these the zero momentum computation was exact, and they have no additional momentum dependence. We note that in some of the diagrams that we do not take into account there are non-vanishing six-derivative contributions which are not m-dependent, and are finite. But here we focus only on the m-dependent terms. Note that when discussing the $m \to 0$ limit one has to be careful, since this limit does not commute with the small momentum limit that we analyze here.

6.4.1 Fermion diagrams

The diagram M^4_{F1} is given by:

$$\begin{aligned}
M^4_{F1} &= (-1) \times \delta_{ij}\delta_{kl} \sum_F \int \frac{d^2p}{(2\pi)^2} (\frac{i}{4T})^2 \frac{(-4i)^2 p_-(p_- - k_-)}{(p^2+m_F^2)((p-k)^2+m_F^2)} \frac{1}{4} \mathrm{tr}[(1+\gamma^c)^2 \tilde{\gamma}_F^T \tilde{\gamma}_F] \frac{1}{2} \times \\
&\quad \{ \ (k_1 \cdot k_2 p_+ - k_{1+} k_2 \cdot p - k_{2+} k_1 \cdot p)(k_3 \cdot k_4 (2p_+ - k_+) - k_{3+} k_4 \cdot (2p - k) - k_{4+} k_3 \cdot (2p-k)) \\
&\quad + (k_3 \cdot k_4 p_+ - k_{3+} k_4 \cdot p - k_{4+} k_3 \cdot p)(k_1 \cdot k_2 (2p_+ - k_+) - k_{1+} k_2 \cdot (2p-k) - k_{2+} k_1 \cdot (2p-k))\} \\
&= \frac{2}{T^2} \delta_{ij}\delta_{kl} \sum_F \int_0^1 d\alpha \int \frac{d^2p}{(2\pi)^2} \frac{\alpha(1-\alpha)}{(p^2+m_F^2+k^2\alpha(1-\alpha))^2} k_- k_- \times \\
&\quad \{k_1 \cdot k_2 k_3 \cdot k_4 p_+ p_+ - k_1 \cdot k_2 (k_3 \cdot p k_{4+} p_+ + k_4 \cdot p k_{3+} p_+) - k_3 \cdot k_4 (k_2 \cdot p k_{1+} p_+ + k_1 \cdot p k_{2+} p_+) \\
&\quad + k_1 \cdot p k_3 \cdot p k_{2+} k_{4+} + k_2 \cdot p k_3 \cdot p k_{1+} k_{4+} + k_1 \cdot p k_4 \cdot p k_{2+} k_{3+} + k_2 \cdot p k_4 \cdot p k_{1+} k_{3+}\} = 0. \qquad (6.27)
\end{aligned}$$

Similarly, we find that $M^4_{F2} = 0$ exactly.

6.4.2 Scalar diagrams

The only scalar diagram that can contribute an m-dependence is M^4_{B1}. We write only the

term proportional to $\delta_{ij}\delta_{kl}$:

$$\begin{aligned}
M^4_{B1} &= 2(\frac{i}{2T})^2 \sum_B \int \frac{d^2p}{(2\pi)^2}[m_B^2 k_1\cdot k_2 + k_1\cdot k_2 p\cdot(p-k) - k_1\cdot p k_2\cdot(p-k) - k_2\cdot p k_1\cdot(p-k)] \\
&\times [m_B^2 k_3\cdot k_4 + k_3\cdot k_4 p\cdot(p-k) - k_3\cdot p k_4\cdot(p-k) - k_4\cdot p k_3\cdot(p-k)]\frac{-i}{p^2+m_B^2}\frac{-i}{(p-k)^2+m_B^2} \\
&= \frac{1}{2T^2}\sum_B \int \frac{d^2p}{(2\pi)^2}[(m_B^2+p^2)k_1\cdot k_2 - 2k_1\cdot p k_2\cdot p] \\
&\times [(m_B^2+p^2)k_3\cdot k_4 - 2k_3\cdot p k_4\cdot p]\frac{1}{p^2+m_B^2}\frac{1}{(p-k)^2+m_B^2} \\
&= \frac{1}{2T^2}\sum_B \int_0^1 d\alpha \int \frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_B^2+k^2\alpha(1-\alpha))^2} \\
&\times [(m_B^2+(p+k(1-\alpha))^2)k_1\cdot k_2 - 2k_1\cdot(p+k(1-\alpha))k_2\cdot(p+k(1-\alpha))] \\
&\times [(m_B^2+(p+k(1-\alpha))^2)k_3\cdot k_4 - 2k_3\cdot(p+k(1-\alpha))k_4\cdot(p+k(1-\alpha))] \\
&= \frac{1}{2T^2}\sum_B \int_0^1 d\alpha \int \frac{d^2p}{(2\pi)^2}[(m_B^2+p^2)k_1\cdot k_2 - 2k_1\cdot p k_2\cdot p] \\
&\times [(m_B^2+p^2)k_3\cdot k_4 - 2k_3\cdot p k_4\cdot p]\frac{1}{(p^2+m_B^2+k^2\alpha(1-\alpha))^2} \\
&= \frac{i}{32\pi T^2}\sum_B [4sm_B^4 \tanh^{-1}[\sqrt{\frac{s}{s+4m_B^2}}](1+\frac{4m_B^2}{s})^{-\frac{1}{2}} \\
&\qquad + \{-s^2+t^2+u^2\}\cdot\{\frac{m_B^2}{2} + \frac{(s+4m_B^2)^2}{3\sqrt{s(s+4m_B^2)}}\tanh^{-1}[\sqrt{\frac{s}{s+4m_B^2}}] \\
&\qquad\qquad + \frac{1}{18}(-5s - 24m_B^2 - 9\Lambda^2 + 3(s+6m_B^2)\log[\frac{m_B^2}{\Lambda^2}])\}]. \qquad (6.28)
\end{aligned}$$

We can check that we reproduce the leading term (6.23) we computed above and we see there is a branch cut at $s = -4m_B^2$, as expected. Taking $k \to 0$, we find up to order k^6

$$M^4_{B1} = \frac{i}{32\pi T^2}\delta_{ij}\delta_{kl}\sum_B \left[m_B^2 s^2 - \frac{1}{6}s^3 + \left\{-m_B^2 - \Lambda^2 + (2m_B^2 + \frac{s}{3})\log[\frac{\Lambda^2}{m_B^2}]\right\} tu\right] \qquad (6.29)$$

We see that there is apparently a logarithmic m-dependent term at six-derivative order, but in fact it is proportional to stu which vanishes on-shell. Thus, at one-loop order we find no corrections to the six-derivative four-scalar term in our effective action. This is not too surprising, since the relevant term is only non-trivial for $D > 3$, while our computation here is essentially independent of D.

6.5 6-point function

In this section we compute the six-point function $\langle X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)X_m(k_5)X_n(k_6)\rangle$. We write here only the scalar contribution to six-derivative terms. We compute the diagrams for zero external momenta, and so we expect the result to be proportional to $\sum_B m_B^2 \log[m_B^2/\Lambda^2]$. We know the fermionic contribution should be $\sum_F m_F^2 \log[m_F^2/\Lambda^2]$,

with the same proportionality constant and opposite sign, in order to have a finite result. Thus we skip the explicit computation of fermionic diagrams, and calculate only the scalar contribution, from which we keep only the term proportional to $m_B^2 \log[m_B^2]$.

As in previous subsections, we focus only on the $\delta_{ij}\delta_{kl}\delta_{mn}$ terms. There are 3 scalar diagrams (see figure 7). The first is :

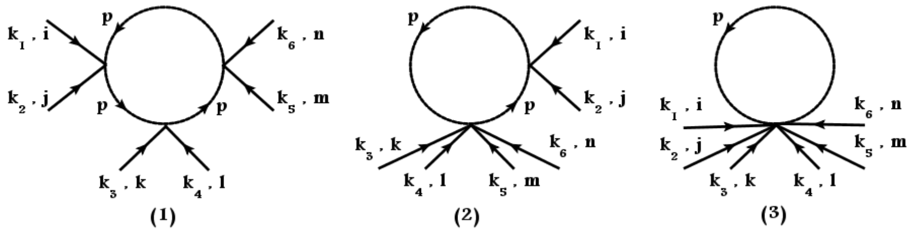

Figure 7: The 6-point scalar diagrams: (1)M^6_{B1}, (2)M^6_{B2}, (3)M^6_{B3}.

$$
\begin{aligned}
M^6_{B1} &= 8 \times (\frac{i}{2T})^3 \sum_B \int \frac{d^2p}{(2\pi)^2} (\frac{-i}{m_B^2 + p^2})^3 (k_1 \cdot k_2 (p^2 + m^2) - 2k_1 \cdot p k_2 \cdot p) \\
&\times (k_3 \cdot k_4 (p^2 + m^2) - 2k_3 \cdot p k_4 \cdot p)(k_5 \cdot k_6 (p^2 + m_B^2) - 2k_5 \cdot p k_6 \cdot p) \\
&= \frac{i}{2\pi T^3} \sum_B \{ k_1 \cdot k_2 k_3 \cdot k_4 k_5 \cdot k_6 \left(-\frac{1}{3}\Lambda^2 + \frac{13}{24} m_B^2 - \frac{1}{4} m_B^2 \log[\frac{\Lambda^2}{m_B^2}] \right) \\
&+ \left(\frac{\Lambda^2}{6} + \frac{m_B^2}{24} + \frac{m_B^2}{4} \log[\frac{m_B^2}{\Lambda^2}] \right) [k_1 \cdot k_2 k_3 \cdot k_5 k_4 \cdot k_6 + k_1 \cdot k_2 k_3 \cdot k_6 k_4 \cdot k_5 \\
&+ (1;2) \leftrightarrow (3;4) + (1;2) \leftrightarrow (5;6)] \\
&- \frac{1}{6} \left(\frac{\Lambda^2}{2} + \frac{5m_B^2}{4} + \frac{3}{2} m_B^2 \log[\frac{m_B^2}{\Lambda^2}] \right) [k_1 \cdot k_3 (k_2 \cdot k_5 k_4 \cdot k_6 + k_2 \cdot k_6 k_4 \cdot k_5) \\
&+ k_1 \cdot k_4 (k_2 \cdot k_5 k_3 \cdot k_6 + k_2 \cdot k_6 k_3 \cdot k_5) + (1;2) \leftrightarrow (3;4) + (1;2) \leftrightarrow (5;6)] \}. \qquad (6.30)
\end{aligned}
$$

The factor of (8) here is a symmetry factor for the possible contractions in the loop. Next,

$$
\begin{aligned}
M^6_{B2} &= 2 \times (\frac{i}{2T})(\frac{i}{2T^2}) \sum_B \int \frac{d^2p}{(2\pi)^2} \frac{(-i)^2}{(p^2 + m_B^2)^2} (k_1 \cdot k_2 (p^2 + m^2) - 2k_1 \cdot p k_2 \cdot p) \\
&\times \{ -2k_3 \cdot k_4 k_5 \cdot p k_6 \cdot p - 2k_5 \cdot k_6 k_3 \cdot p k_4 \cdot p + 3k_3 \cdot k_4 k_5 \cdot k_6 p^2 \\
&- p^2 (k_3 \cdot k_5 k_4 \cdot k_6 + k_3 \cdot k_6 k_4 \cdot k_5) + m_B^2 (-k_3 \cdot k_4 k_5 \cdot k_6 + k_3 \cdot k_5 k_4 \cdot k_6 + k_4 \cdot k_5 k_3 \cdot k_6) \} \\
&= \frac{i}{2T^3} \sum_B \int \frac{d^2p}{(2\pi)^2} \frac{1}{(p^2 + m_B^2)^2} \\
&\times \{ k_1 \cdot k_2 k_3 \cdot k_4 k_5 \cdot k_6 [(p^2 + m_B^2)(p^2 (3 - \frac{4}{d}) - m_B^2) + \frac{2}{d} p^2 m_B^2 - 2p^4 (-\frac{4}{d(d+2)} + \frac{3}{d})] \\
&+ k_1 \cdot k_2 (k_3 \cdot k_5 k_4 \cdot k_6 + k_3 \cdot k_6 k_4 \cdot k_5) [(p^2 + m_B^2)(-p^2 + m_B^2) + \frac{2}{d} p^2 (-m_B^2 + p^2)] \\
&+ k_3 \cdot k_4 (k_1 \cdot k_5 k_2 \cdot k_6 + k_1 \cdot k_6 k_2 \cdot k_5) (\frac{4}{d(d+2)} p^4) \\
&+ k_5 \cdot k_6 (k_1 \cdot k_3 k_2 \cdot k_4 + k_1 \cdot k_4 k_2 \cdot k_3) (\frac{4}{d(d+2)} p^4) \} \quad . \qquad (6.31)
\end{aligned}
$$

There is a symmetry factor of (2) for internal contractions. We should now sum over the two permutations $(k_1, k_2) \to (k_3, k_4)$ and $(k_1, k_2) \to (k_5, k_6)$, and then we get

$$
\begin{aligned}
M^6_{B2} &= \frac{1}{2T^3}\sum_B \int \frac{d^2p}{(2\pi)^2}\frac{1}{(p^2+m_B^2)^2} \\
&\times \{3k_1\cdot k_2 k_3\cdot k_4 k_5\cdot k_6[(p^2+m_B^2)(p^2(3-\frac{4}{d})-m_B^2)+\frac{2}{d}p^2m_B^2-2p^4(-\frac{4}{d(d+2)}+\frac{3}{d})] \\
&+ [k_1\cdot k_2(k_3\cdot k_5 k_4\cdot k_6+k_3\cdot k_6 k_4\cdot k_5)+k_3\cdot k_4(k_1\cdot k_5 k_2\cdot k_6+k_1\cdot k_6 k_2\cdot k_5)+(1;2)\leftrightarrow(3;4) \\
&+ (1;2)\leftrightarrow(5;6)]\times[(p^2+m_B^2)(-p^2+m_B^2)+\frac{2}{d}p^2(-m_B^2+p^2)+\frac{8}{d(d+2)}p^4]\} \\
&= \frac{i}{4\pi T^3}\sum_B\left[\frac{\Lambda^2}{2}+\frac{3}{2}m_B^2+\frac{3}{2}m_B^2\log[\frac{m_B^2}{\Lambda^2}]\right]\{-3k_1\cdot k_2k_3\cdot k_4k_5\cdot k_6 \\
&+ [k_1\cdot k_2(k_3\cdot k_5k_4\cdot k_6+k_3\cdot k_6k_4\cdot k_5)+k_3\cdot k_4(k_1\cdot k_5k_2\cdot k_6+k_1\cdot k_6k_2\cdot k_5) \\
&+ (1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\} \quad .
\end{aligned}
\tag{6.32}
$$

Finally,

$$
\begin{aligned}
M^6_{B3} &= \frac{i}{2T^3}\sum_B\int\frac{d^2p}{(2\pi)^2}\frac{-i}{p^2+m_B^2} \\
&\times \{k_1\cdot k_2k_3\cdot k_4k_5\cdot k_6(9p^2-3m^2)-2k_1\cdot pk_2\cdot pk_3\cdot k_4k_5\cdot k_6 \\
&- 2k_3\cdot pk_4\cdot pk_1\cdot k_2k_5\cdot k_6-2k_5\cdot pk_6\cdot pk_3\cdot k_4k_1\cdot k_2 \\
&+ (m^2-p^2)[k_1\cdot k_2(k_3\cdot k_5k_4\cdot k_6+k_3\cdot k_6k_4\cdot k_5)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)] \\
&- 2[k_1\cdot pk_2\cdot p(k_3\cdot k_5k_4\cdot k_6+k_3\cdot k_6k_4\cdot k_5)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\} \\
&= \frac{1}{2\pi T^3}\sum_B\left(\frac{1}{2}\Lambda^2+\frac{3}{4}m_B^2\log[\frac{m_B^2}{\Lambda^2}]\right)\{3k_1\cdot k_2k_3\cdot k_4k_5\cdot k_6 \\
&- [k_1\cdot k_2(k_3\cdot k_5k_4\cdot k_6+k_3\cdot k_6k_4\cdot k_5)+(1;2)\leftrightarrow(3;4)+(1;2)\leftrightarrow(5;6)]\} \quad .
\end{aligned}
\tag{6.33}
$$

Summing our results, and putting in the necessary fermionic contributions to cancel the divergences, we obtain the following 6-point function,

$$
\begin{aligned}
&\langle X_i(k_1)X_j(k_2)X_k(k_3)X_l(k_4)X_m(k_5)X_n(k_6)\rangle = \frac{i\Delta T}{T^3}\delta_{ij}\delta_{kl}\delta_{mn} \\
&\{k_1\cdot k_2k_3\cdot k_4k_5\cdot k_6-[k_1\cdot k_2(k_3\cdot k_5k_4\cdot k_6+k_4\cdot k_6k_3\cdot k_5)+(k_1,k_2)\to(k_3,k_4) \\
&+(k_1,k_2)\to(k_5,k_6)]+[k_1\cdot k_3(k_2\cdot k_5k_4\cdot k_6+k_2\cdot k_6k_4\cdot k_5) \\
&+k_1\cdot k_4(k_2\cdot k_5k_3\cdot k_6+k_2\cdot k_6k_3\cdot k_5)+(k_1,k_2)\to(k_3,k_4)+(k_1,k_2)\to(k_5,k_6)]\} \\
&+\text{ permutations on }[(i,1),(j,2),(k,3),(l,4),(m,5),(n,6)]\} \quad .
\end{aligned}
\tag{6.34}
$$

However, it turns out that when we use the on-shell constraints this exactly vanishes. Thus, the terms with six derivatives and six X's in our action precisely agree with their Nambu-Goto values, as expected from our general arguments of section 3.

7. Conclusions

In this paper we analyzed the low-energy effective action of confining strings. We computed its partition function using a zeta-function regularization, argued in [9] to be the unique

regularization which gives results that are independent of the UV cutoff (as we expect). We showed that up to four-derivative order this action must agree with the Nambu-Goto form, generalizing a result of Lüscher and Weisz for $D = 3$. At the six-derivative order there are three possible terms for general D, and we showed that our considerations do not constrain the term c_4 that does not appear in the Nambu-Goto action (the two terms appearing in Nambu-Goto, c_6 and c_7, are uniquely determined). Somewhat surprisingly, we found that this coefficient does not contribute to the partition function on the torus at the first possible order, corresponding to corrections to closed string energies of order $1/L^5$. Thus, the corrections to energy levels coming from this term must sum to zero separately at each energy level. In particular we claim that the closed string ground state energy is not corrected at order $1/L^5$ (compared to the Nambu-Goto result), so its first corrections arise at least at order $1/L^7$. For the special case of $D = 3$ we find that there is only one coefficient in the effective action at six-derivative order, which is uniquely determined, so that all energy levels must agree with the Nambu-Goto results up to order $1/L^5$. This seems to be consistent with lattice results indicating that the corrections to Nambu-Goto for the ground state are very small [27, 28]; it is not consistent with lattice results for percolation presented in [33], but it is not clear if these results are reliable and if the corresponding string theory has a weakly coupled limit where our results should apply. It would be interesting to use lattice simulations to measure the value of c_4 for interesting confining theories with $D > 3$, such as the pure Yang-Mills theory in $D = 4$.

In the partition function on the annulus, we found that the c_4 term does contribute corrections to closed string energies when $D > 3$. Recall that while the torus partition function sums over all closed string states with weight one, the annulus partition function sums only over specific closed string states, which have some overlap with the boundary state, and the sum comes with different coefficients for different states.

We then computed the specific coefficients appearing in the effective action in a large class of holographic backgrounds, by integrating out the massive fields on the worldsheet of strings in confining backgrounds, to leading order in the curvature. We verified that up to four-derivatives the Nambu-Goto action is reproduced, and we showed that also at six-derivative order the effective action precisely agrees with Nambu-Goto. Somewhat surprisingly, we did not find any c_4 term; it is possible that such a term only arises at higher orders, or that it is constrained to vanish by considerations different than the ones we used here (for instance, by the constraints arising in the formalism of Polchinski and Strominger). It would be interesting to understand this better. In any case, in the backgrounds we study this means that at one-loop order there is no correction to the partition function at six-derivative order, both on the torus and on the annulus (at least with the specific boundary terms we chose).

Some possible generalizations of our analysis are :

- It would be interesting to go to higher orders in the derivative expansion, in particular to see at what order the effective action for $D = 3$ can first deviate from the Nambu-Goto form, and at what order corrections to the ground state energy (for any D) can start appearing.

- It would be interesting to go to two-loop order in the computation of the effective action in holographic backgrounds, to see whether the c_4 term is generated at this order or not. In particular, it would be interesting to see whether the effective action computed in this formalism precisely agrees with the Nambu-Goto action (to all orders in the derivative expansion), or whether deviations occur at some order, and, if so, at what order the deviations first occur.

- We computed the effective action and the resulting partition function, but we did not use the effective action to compute the corrections to specific energy levels; it would be interesting to do this.

- We focused on closed strings, and assumed there are no boundary terms. It would be interesting to analyze what are the possible boundary terms that could contribute up to the order we worked in, and to compute the corresponding corrections to open string energies. In particular, it would be interesting to see which boundary terms appear in the computation of the quark-anti-quark potential in holographic backgrounds (see [40] for the expansion of the action of a holographic open confining string to quadratic order in fluctuations).

- We assumed that the only massless fields on the worldsheet are the transverse fluctuations, but in many interesting cases (like supersymmetric gauge theories) there are additional massless fields on the worldsheet. It would be interesting to generalize our analysis to these cases, to see what are the allowed terms in the effective action and whether they contribute to the partition function or not.

- As we mentioned in section 2, our analysis does not apply directly to k-strings since they have light states on their worldsheet in the large N limit; it would be interesting to understand better the form of the low-energy effective action on k-strings, and to match it with recent lattice results.

- We considered here only orientable strings, which are relevant for $SU(N)$ gauge theories. For $SO(N)$ or $USp(N)$ theories the confining string is unoriented, so additional diagrams (such as a Klein bottle) are possible. It would be interesting to check if these diagrams give additional constraints on the effective action, and to compute the effective action for holographic backgrounds that correspond to such theories.

- In our analysis we wrote the effective action using only the physical transverse fluctuations. It would be interesting to compare this, and our constraints on the possible terms, to other formalisms, such as working in a Poincaré invariant formalism and adding terms involving the extrinsic curvature of the worldsheet (such as the "rigidity term" [41, 42]), and the Polchinski-Strominger approach [2]. The standard "rigidity term" seems to be trivial in our long-string effective action (in the sense that we can get rid of it by field definitions) up to the order that we work in, but it may appear at higher orders.

- It may also be possible to obtain inequalities on coefficients in the effective action by using unitarity considerations, as (for instance) in [43]. For example, the positivity of c_2 (which is true) may be argued just from these considerations. It would be interesting to incorporate these additional constraints into our analysis.

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A. Computations for section 3

We follow the computation technique used in [9], involving zeta function regularization, and we perform our calculations in position space. The Feynman bubble diagrams appearing in the partition function computations will be sums over the modes of the worldsheet fields. These sums are typically UV divergent, and must be regularized. Fortunately, the regularization scheme used in [9] is claimed to produce a unique finite result for any calculation which is not cutoff-dependent. We know the partition function is finite. Furthermore, we expect it to be finite diagram by diagram (in the low-energy effective action), since in our full action the divergences cancel between scalars and fermions (while the low-energy effective action includes only scalars). The diagrams should also fulfill requirements such as scale invariance and other requirements which are listed in [9]. The claim is that for such a calculation, there is a unique finite result, which is obtained using the regularization scheme of [9].

In this scheme we analytically continue the sums using the zeta function, so that a generic sum is written as

$$\sum_{n,m=-\infty}^{\infty} \frac{m^a n^b}{m^2+n^2} \equiv \sum_{n,m=-\infty}^{\infty} \frac{m^{a+s} n^{b+s'}}{m^2+n^2}|_{s,s'=0} \ . \tag{A.1}$$

This can be further manipulated such that the divergent part will always appear as a zeta function, as we will see in subsection A.2. The difference of this scheme from dimensional regularization is that there is no single parameter which we perform the analytic continuation in. We analytically continue and regularize each sum by itself. This will give us finite results as long as we do not hit any poles of the zeta function, that is as long as we do not have logarithmically divergent sums ($\zeta(1)$). This is similar to dimensional regularization, where only logarithmic divergences are seen.

In the first two subsections we define some modular functions and compute divergent sums which appear in our computation. We explicitly write their regularization using the zeta function. In the following subsections we write the details of the partition function computation at order $O(T^{-1})$ and $O(T^{-2})$, both on the annulus and on the torus. Finally, we explain the numerical method used to determine the $\tilde{q}$ expansion of $F(q)$.

A.1 Modular functions

Below is a list of functions which are related to the Dedekind eta function (3.4), and have nice properties under modular transformations. We define $q, \tilde{q}, \tau, \tilde{\tau}$ as in (3.2). We recall the Eisenstein series $E_{2k}(q)$,

$$E_{2k}(q) = 1 + \frac{2}{\zeta(1-2k)} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n}, \tag{A.2}$$

and define the functions $H_{2,k}(q)$ (for even k) :

$$H_{2,k}(q) \equiv \frac{\zeta(1-k)}{2} q \frac{\partial}{\partial q} E_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{(1-q^n)^2}. \tag{A.3}$$

For the cases we will encounter these are given by

$$H_{2,2}(q) = \frac{E_4(q) - E_2(q)^2}{288}, \qquad H_{2,4}(q) = \frac{E_2(q)E_4(q) - E_6(q)}{720}. \tag{A.4}$$

These functions obey the modular transformation properties :

$$\begin{aligned} E_2(q) &= -\frac{6i}{\pi}\tilde{\tau} + \tilde{\tau}^2 E_2(\tilde{q}), \quad H_{2,2}(q) = \frac{\log(\tilde{q})^2}{4\pi^4}[-\frac{1}{8} - \frac{1}{48}\log(\tilde{q})E_2(\tilde{q}) + \frac{1}{4}\log(\tilde{q})^2 H_{2,2}(\tilde{q})], \\ E_k(q) &= \tilde{\tau}^k E_k(\tilde{q}), \quad H_{2,k}(q) = -\frac{ik\zeta(1-k)}{4\pi}\tilde{\tau}^{k+1}E_k(\tilde{\tau}) + \tilde{\tau}^{k+2}H_{2,k}(\tilde{q}) \quad \forall k > 2. \end{aligned} \tag{A.5}$$

Finally, we define the function $F(q^{ann.})$:

$$\begin{aligned} F(q^{ann.}) &= \sum_{n,r=1}^{\infty} nr(n+r)\coth(\frac{n\pi L}{2R})\coth(\frac{(n+r)\pi L}{2R})\coth(\frac{r\pi L}{2R}), \\ &= \sum_{r>n,n=1}^{\infty} nr(n-r)\coth(\frac{n\pi L}{2R})\coth(\frac{(n-r)\pi L}{2R})\coth(\frac{r\pi L}{2R}) \quad . \end{aligned} \tag{A.6}$$

A.2 Regularization of sums

Below we list the sums which appear in our computations. We write the sums appearing in the annulus computation, but they are simply related to the ones appearing for the torus. The list includes diverging sums, which we manipulate such that the divergence is always

expressed using a zeta function.

$$\sum_{m=-\infty}^{\infty}\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}} = \frac{\pi RL}{2n}\coth(\frac{n\pi L}{2R}),\quad \sum_{m=1}^{\infty}\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}} = \frac{\pi RL}{4n}\coth(\frac{n\pi L}{2R})-\frac{R^2}{2n^2}$$
$$\sum_{m=-\infty}^{\infty}\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}} = \sum_{m=-\infty}^{\infty}\frac{m^{2+s}}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}}|_{s=0} = \frac{L^2}{4}\sum_{m=-\infty}^{\infty}m^s(1-\frac{n^2}{R^2}\frac{1}{\frac{n^2m^s}{R^2}+\frac{4m^2}{L^2}})|_{s=0}$$
$$= \frac{L^2}{4}(1+2\zeta(0))-\frac{L^3n\pi}{8R}\coth(\frac{\pi nL}{2R}) = -\frac{L^3n\pi}{8R}\coth(\frac{\pi nL}{2R}),$$
$$\sum_{m=-\infty}^{\infty}\frac{m^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}} = \sum_{m=-\infty}^{\infty}\frac{m^{4+s}}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}}|_{s=0} = \frac{L^2}{4}\sum_{m=-\infty}^{\infty}m^{2+s}(1-\frac{n^2}{R^2}\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})|_{s=0}$$
$$= \frac{L^2}{2}\zeta(-2)-\frac{L^2n^2}{4R^2}\sum_{m=-\infty}^{\infty}\frac{m^{2+s}}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}} = \frac{L^5n^3\pi}{32R^3}\coth(\frac{n\pi L}{2R}),$$

$$\sum_{n=1}^{\infty}n^s\coth(\frac{n\pi L}{2R}) = \sum_{n=1}^{\infty}n^s\frac{e^{\frac{\pi nL}{R}}+1}{e^{\frac{\pi nL}{R}}-1} = \sum_{n=1}^{\infty}n^s(1+\frac{2}{e^{\frac{\pi nL}{R}}-1}) = \zeta(-s)+2\sum_{n=1}^{\infty}\frac{n^s q_{ann.}^n}{1-q_{ann.}^n}$$
$$= \zeta(-s)E_{s+1}(q^{ann.}),$$
$$\sum_{n=1}^{\infty}n^s\coth^2(\frac{n\pi L}{2R}) = \sum_{n=1}^{\infty}n^s\left(\frac{e^{\frac{\pi nL}{R}}+1}{e^{\frac{\pi nL}{R}}-1}\right)^2 = \sum_{n=1}^{\infty}n^s\left(1+4\frac{e^{\frac{\pi nL}{R}}}{(e^{\frac{\pi nL}{R}}-1)^2}\right)$$
$$= \zeta(-s)+4\sum_{n=1}^{\infty}\frac{n^s q_{ann.}^n}{(1-q_{ann.}^n)^2} = \zeta(-s)+4H_{2,s}(q^{ann.}). \qquad \text{(A.7)}$$

A.3 The annulus

The Green's function on a cylinder worldsheet with rectangular domain (R, L) is

$$G(\sigma_1,\sigma_0;\sigma_1{}',\sigma_0{}') = \frac{2}{\pi^2 RL}\sum_{n=1,m=-\infty}^{\infty}\frac{\sin(\frac{n\pi\sigma_1}{R})\sin(\frac{n\pi\sigma_1{}'}{R})e^{\frac{2\pi i m(\sigma_0-\sigma_0{}')}{L}}}{\frac{n^2}{R^2}+4\frac{m^2}{L^2}}\quad . \qquad \text{(A.8)}$$

Here L is the periodic direction, and $0 \le \sigma_1 \le R$.

A.3.1 Partition function at $O(L^{-3})$

This computation was already performed in [9], and we reproduce it here for completeness. The diagrams at this order are (see figure 1)

$$I_1^{ann.} = \int d^2\sigma\partial_\alpha\partial^{\alpha'}G\partial_\beta\partial^{\beta'}G = \int d^2\sigma\{(\partial_0\partial_0'G+\partial_1\partial_1'G)(\partial_0\partial_0'G+\partial_1\partial_1'G)\}$$
$$= \frac{2\pi^2 L}{R^3}H_{2,2}(q^{ann.}) = -\frac{1}{LR}+\frac{2\pi}{3L^2}E_2(\tilde{q})+\frac{32\pi^2 R}{L^3}H_{2,2}(\tilde{q}),$$
$$I_2^{ann.} = \int d^2\sigma\partial_\alpha\partial_{\beta'}G\partial^\alpha\partial^{\beta'}G = \int d^2\sigma\{\partial_0\partial_0'G\partial_0\partial_0'G+\partial_1\partial_1'G\partial_1\partial_1'G\}$$
$$= \frac{\pi^2 L}{R^3}\left[\frac{2}{(24)^2}E_2^2(q^{ann.})+H_{2,2}(q^{ann.})\right] = \frac{16\pi^2 R}{L^3}\left[\frac{2}{(24)^2}E_2^2(\tilde{q})+H_{2,2}(\tilde{q})\right], \qquad \text{(A.9)}$$

where

$$\partial_\alpha = \frac{\partial}{\partial \sigma_\alpha}, \qquad \partial_\alpha \partial'_\alpha G = \lim_{\sigma' \to \sigma} \partial_\alpha \partial'_\alpha G(\sigma, \sigma') \quad . \tag{A.10}$$

In this notation, we first take the derivative with respect to σ or σ' and only then take the limit $\sigma \to \sigma'$. One should notice that an odd number of derivatives of the propagators with respect to σ^0 gives a sum of antisymmetric functions of $\sigma^0 - \sigma^{0\prime}$, and therefore vanishes as $\sigma^0 \to \sigma^{0\prime}$.

Below are the details of the computation:

$$\begin{aligned}
&\int d^2\sigma \partial_1 \partial'_1 G \partial_1 \partial'_1 G \\
&= \frac{4}{R^6 L^2} \sum_{n,k=1}^{\infty} \sum_{m,l=-\infty}^{\infty} \frac{n^2 k^2}{(\frac{n^2}{R^2} + \frac{4m^2}{L^2})(\frac{k^2}{R^2} + \frac{4l^2}{L^2})} \int d^2\sigma \cos^2(\frac{n\pi\sigma}{R}) \cos^2(\frac{k\pi\sigma}{R}) \\
&= \frac{1}{R^5 L} \{ (\sum_{m,n} \frac{n^2}{\frac{n^2}{R^2} + \frac{4m^2}{L^2}})(\sum_{k,l} \frac{k^2}{\frac{k^2}{R^2} + \frac{4l^2}{L^2}}) + \frac{1}{2} \sum_n n^4 (\sum_m \frac{1}{\frac{n^2}{R^2} + \frac{4m^2}{L^2}})(\sum_l \frac{1}{\frac{n^2}{R^2} + \frac{4l^2}{L^2}}) \} \\
&= \frac{\pi^2 L}{4R^3} \left((\zeta(-1) E_2(q^{ann.}))^2 + \frac{1}{2} (\zeta(-2) + 4H_{2,2}(q^{ann.})) \right),
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
&\int d^2\sigma \partial_0 \partial'_0 G \partial_1 \partial'_1 G \\
&= \frac{16}{R^4 L^4} \sum_{n,k=1}^{\infty} \sum_{m,l=-\infty}^{\infty} \frac{n^2 l^2}{(\frac{n^2}{R^2} + \frac{4m^2}{L^2})(\frac{k^2}{R^2} + \frac{4l^2}{L^2})} \int d^2\sigma \cos^2(\frac{n\pi\sigma}{R}) \sin^2(\frac{k\pi\sigma}{R}) \\
&= \frac{4}{R^3 L^3} \{ (\sum_{m,n} \frac{n^2}{\frac{n^2}{R^2} + \frac{4m^2}{L^2}})(\sum_{k,l} \frac{l^2}{\frac{k^2}{R^2} + \frac{4l^2}{L^2}}) - \frac{1}{2} \sum_n (\sum_m \frac{n^2}{\frac{n^2}{R^2} + \frac{4m^2}{L^2}})(\sum_l \frac{l^2}{\frac{n^2}{R^2} + \frac{4l^2}{L^2}}) \} \\
&= -\frac{\pi^2 L}{4R^3} \left((\zeta(-1) E_2(q^{ann.}))^2 - \frac{1}{2} (\zeta(-2) + 4H_{2,2}(q^{ann.})) \right),
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
&\int d^2\sigma \partial_0 \partial'_0 G \partial_0 \partial'_0 G \\
&= \frac{64}{R^2 L^6} \sum_{n,k=1}^{\infty} \sum_{m,l=-\infty}^{\infty} \frac{m^2 l^2}{(\frac{n^2}{R^2} + \frac{4m^2}{L^2})(\frac{k^2}{R^2} + \frac{4l^2}{L^2})} \int d^2\sigma \sin^2(\frac{n\pi\sigma}{R}) \sin^2(\frac{k\pi\sigma}{R}) \\
&= \frac{16}{R L^5} \{ (\sum_{m,n} \frac{m^2}{\frac{n^2}{R^2} + \frac{4m^2}{L^2}})(\sum_{k,l} \frac{l^2}{\frac{k^2}{R^2} + \frac{4l^2}{L^2}}) + \frac{1}{2} \sum_n (\sum_m \frac{m^2}{\frac{n^2}{R^2} + \frac{4m^2}{L^2}})(\sum_l \frac{l^2}{\frac{n^2}{R^2} + \frac{4l^2}{L^2}}) \} \\
&= \frac{\pi^2 L}{4R^3} \left((\zeta(-1) E_2(q^{ann.}))^2 + \frac{1}{2} (\zeta(-2) + 4H_{2,2}(q^{ann.})) \right) \quad .
\end{aligned} \tag{A.13}$$

A.3.2 Partition function at $O(L^{-5})$

At this order there are both 2-loop and 3-loop diagram contributions (see figures 1 and 2).

At two loops there are two possible contractions,

$$
\begin{aligned}
I_3^{ann.} &= \int d^2\sigma \partial_\alpha \partial^{\alpha'} \partial_\beta \partial^{\beta'} G \partial_\gamma \partial^{\gamma'} G \\
&= \int d^2\sigma (\partial_0\partial_0'\partial_0\partial_0' G + \partial_1\partial_1'\partial_1\partial_1' G + 2\partial_1\partial_1'\partial_0\partial_0' G)(\partial_1\partial_1' G + \partial_0\partial_0' G) \\
&= -4\frac{\pi^4 L}{R^5} H_{2,4}(q^{ann.}) = -4\frac{\pi^4 L}{R^5}\left(\frac{4R^5}{15\pi L^5}E_4(\tilde{\tau}) - \frac{64R^6}{L^6}H_{2,4}(\tilde{q})\right), \qquad \text{(A.14)} \\
I_4^{ann.} &= \int d^2\sigma \partial_\alpha \partial_\beta \partial_{\gamma'} G \partial^\alpha \partial^\beta \partial^{\gamma'} G \\
&= \int d^2\sigma \{\partial_0\partial_0\partial_0' G \partial_0\partial_0\partial_0' G + 2\partial_0\partial_1\partial_1' G \partial_0\partial_1\partial_1' G + \partial_0'\partial_1\partial_1 G \partial_0'\partial_1\partial_1 G \\
&\quad +2\partial_1\partial_0\partial_0' G \partial_1\partial_0\partial_0' G + \partial_0\partial_0\partial_1' G \partial_0\partial_0\partial_1' G + \partial_1\partial_1\partial_1' G \partial_1\partial_1\partial_1' G\} \\
&= 2\frac{\pi^4 L}{R^5} H_{2,4}(q^{ann.}). \qquad \text{(A.15)}
\end{aligned}
$$

Below are the details of the computation:

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1'\partial_1\partial_1' G \partial_1\partial_1' G \\
&= \frac{4\pi^2}{R^8 L^2}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^4 k^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \sin^2(\frac{n\pi\sigma}{R})\cos^2(\frac{k\pi\sigma}{R}) \\
&= \frac{\pi^2}{R^7 L}\{(\sum_{m,n}\frac{n^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}}) - \frac{1}{2}\sum_n n^6(\sum_m \frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l \frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^4 L}{4R^5}(\zeta(-3)\zeta(-1)E_2(q^{ann.})E_4(q^{ann.}) - \frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))), \qquad \text{(A.16)}
\end{aligned}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1'\partial_1\partial_1 G \partial_1\partial_1' G \\
&= -\frac{4\pi^2}{R^8 L^2}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^4 k^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \cos^2(\frac{n\pi\sigma}{R})\cos^2(\frac{k\pi\sigma}{R}) \\
&= -\frac{\pi^2}{R^7 L}\{(\sum_{m,n}\frac{n^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}}) + \frac{1}{2}\sum_n n^6(\sum_m \frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l \frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= -\frac{\pi^4 L}{4R^5}(\zeta(-3)\zeta(-1)E_2(q^{ann.})E_4(q^{ann.}) + \frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))), \qquad \text{(A.17)}
\end{aligned}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1'\partial_1\partial_1' G \partial_0\partial_0' G \\
&= \frac{16\pi^2}{R^6 L^4}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^4 l^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \sin^2(\frac{n\pi\sigma}{R})\sin^2(\frac{k\pi\sigma}{R}) \\
&= \frac{4\pi^2}{R^5 L^3}\{(\sum_{m,n}\frac{n^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}}) + \frac{1}{2}\sum_n n^4(\sum_m \frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l \frac{l^2}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^4 L}{4R^5}\{-\zeta(-1)\zeta(-3)E_2(q^{ann.})E_4(q^{ann.}) - \frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))\}, \qquad \text{(A.18)}
\end{aligned}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1'\partial_0\partial_0' G \partial_1\partial_1' G \\
&= \frac{16\pi^2}{R^6L^4}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^2m^2k^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \cos^2(\frac{n\pi\sigma}{R})\cos^2(\frac{k\pi\sigma}{R}) \\
&= \frac{4\pi^2}{R^5L^3}\{(\sum_{m,n}\frac{n^2m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})+\frac{1}{2}\sum_n n^4(\sum_m\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^4 L}{4R^5}\{-\zeta(-1)\zeta(-3)E_2(q^{ann.})E_4(q^{ann.})-\frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))\},
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1\partial_0\partial_0' G \partial_1\partial_1' G \\
&= -\frac{16\pi^2}{R^6L^4}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^2m^2k^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \sin^2(\frac{n\pi\sigma}{R})\cos^2(\frac{k\pi\sigma}{R}) \\
&= -\frac{4\pi^2}{R^5L^3}\{(\sum_{m,n}\frac{n^2m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})-\frac{1}{2}\sum_n n^4(\sum_m\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^4 L}{4R^5}\{\zeta(-1)\zeta(-3)E_2(q^{ann.})E_4(q^{ann.})-\frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))\},
\end{aligned}
\tag{A.20}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1'\partial_0\partial_0' G \partial_0\partial_0' G = -\int d^2\sigma \partial_1\partial_1'\partial_0\partial_0 G \partial_0\partial_0' G \\
&= \frac{64\pi^2}{R^4L^6}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^2m^2l^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \cos^2(\frac{n\pi\sigma}{R})\sin^2(\frac{k\pi\sigma}{R}) \\
&= \frac{16\pi^2}{R^3L^5}\{(\sum_{m,n}\frac{n^2m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})-\frac{1}{2}\sum_n n^2(\sum_m\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{l^2}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^4 L}{4R^5}\{\zeta(-1)\zeta(-3)E_2(q^{ann.})E_4(q^{ann.})-\frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))\},
\end{aligned}
\tag{A.21}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_0\partial_0'\partial_0\partial_0' G \partial_1\partial_1' G \\
&= \frac{64\pi^2}{R^4L^6}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{m^4k^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma \cos^2(\frac{n\pi\sigma}{R})\sin^2(\frac{k\pi\sigma}{R}) \\
&= \frac{16\pi^2}{R^3L^5}\{(\sum_{m,n}\frac{m^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})-\frac{1}{2}\sum_n n^2(\sum_m\frac{m^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^4 L}{4R^5}\{\zeta(-1)\zeta(-3)E_2(q^{ann.})E_4(q^{ann.})-\frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))\},
\end{aligned}
\tag{A.22}
$$

$$\int d^2\sigma \partial_0\partial_0'\partial_0\partial_0' G\partial_0\partial_0' G = -\int d^2\sigma\partial_0\partial_0'\partial_0\partial_0 G\partial_0\partial_0' G$$

$$= \frac{256\pi^2}{R^2L^8}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{m^4l^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\int d^2\sigma\sin^2(\frac{n\pi\sigma}{R})\sin^2(\frac{k\pi\sigma}{R})$$

$$= \frac{64\pi^2}{RL^7}\{(\sum_{m,n}\frac{m^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})+\frac{1}{2}\sum_n(\sum_m\frac{m^4}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{l^2}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\}$$

$$= \frac{\pi^4L}{4R^5}\{-\zeta(-1)\zeta(-3)E_2(q^{ann.})E_4(q^{ann.})-\frac{1}{2}(\zeta(-4)+4H_{2,4}(q^{ann.}))\}, \tag{A.23}$$

$$\int d^2\sigma\partial_1\partial_1\partial_1 G\partial_1'\partial_1'\partial_1' G = \int d^2\sigma\partial_1'\partial_1\partial_1 G\partial_1'\partial_1\partial_1 G \tag{A.24}$$

$$= \frac{4\pi^2}{R^8L^2}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^3k^3}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\frac{1}{4}\int d^2\sigma\sin(\frac{2n\pi\sigma}{R})\sin(\frac{2k\pi\sigma}{R})$$

$$= \frac{\pi^2}{2R^7L}\{\sum_n n^6(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} = \frac{\pi^4L}{8R^5}(\zeta(-4)+4H_{2,4}(q^{ann.})),$$

$$\int d^2\sigma\partial_0\partial_0\partial_1 G\partial_0'\partial_0'\partial_1' G = \int d^2\sigma\partial_0\partial_0'\partial_1 G\partial_0\partial_0'\partial_1 G = -\int d^2\sigma\partial_0\partial_0'\partial_1 G\partial_0\partial_0\partial_1' G$$

$$= \frac{64\pi^2}{R^4L^6}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{nkm^2l^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}\frac{1}{4}\int d^2\sigma\sin(\frac{2n\pi\sigma}{R})\sin(\frac{2k\pi\sigma}{R}) \tag{A.25}$$

$$= \frac{8\pi^2}{R^3L^5}\{\sum_n n^2(\sum_m\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{l^2}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} = \frac{\pi^4L}{8R^5}(\zeta(-4)+4H_{2,4}(q^{ann.})),$$

$$\int d^2\sigma\partial_1\partial_1'\partial_1 G\partial_1'\partial_0\partial_0' G = \int d^2\sigma\partial_1\partial_1'\partial_1 G\partial_1\partial_0\partial_0' G \tag{A.26}$$

$$= \frac{16\pi^2}{R^6L^4}\sum_{n,k=1}^{\infty}\sum_{m,l=-\infty}^{\infty}\frac{n^3kl^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})}(-\frac{1}{4})\int d^2\sigma\sin(\frac{2n\pi\sigma}{R})\sin(\frac{2k\pi\sigma}{R})$$

$$= \frac{2\pi^2}{R^5L^3}\{-\sum_n n^4(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{l^2}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})\} = \frac{\pi^4L}{8R^5}(\zeta(-4)+4H_{2,4}(q^{ann.})) \quad .$$

At three loops there are 3 possible contractions,

$$I_6^{ann.} = \int d^2\sigma\{\partial_0\partial_0' G\partial_0\partial_0' G\partial_0\partial_0' G+\partial_1\partial_1' G\partial_1\partial_1' G\partial_1\partial_1' G$$
$$+3\partial_0\partial_0' G\partial_0\partial_0' G\partial_1\partial_1' G+3\partial_1\partial_1' G\partial_1\partial_1' G\partial_0\partial_0' G\} = \frac{3\pi^3L}{2R^5}F(q^{ann.}), \tag{A.27}$$

$$I_7^{ann.} = \int d^2\sigma\partial^\alpha\partial_\alpha' G\partial^\beta\partial_\gamma' G\partial_\beta'\partial^\gamma G = \int d^2\sigma\{\partial_0\partial_0' G\partial_0\partial_0' G\partial_0\partial_0' G+\partial_1\partial_1' G\partial_1\partial_1' G\partial_1\partial_1' G$$
$$+\partial_0\partial_0' G\partial_0\partial_0' G\partial_1\partial_1' G+\partial_1\partial_1' G\partial_1\partial_1' G\partial_0\partial_0' G\} = \frac{3\pi^3L}{4R^5}F(q^{ann.}), \tag{A.28}$$

$$I_8^{ann.} = \int d^2\sigma\partial^\alpha\partial_\beta' G\partial^\beta\partial_\gamma' G\partial^\gamma\partial_{\alpha'} G = \int d^2\sigma\{\partial_0\partial_0' G\partial_0\partial_0' G\partial_0\partial_0' G+\partial_1\partial_1' G\partial_1\partial_1' G\partial_1\partial_1' G\}$$
$$= \frac{3\pi^3L}{8R^5}F(q^{ann.}). \tag{A.29}$$

The details are :

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1' G \partial_1\partial_1' G \partial_1\partial_1' G \\
&= \frac{8}{R^9L^3}\sum_{n,k,r=1}^{\infty}\sum_{m,l,s=-\infty}^{\infty}\frac{n^2k^2r^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})(\frac{r^2}{R^2}+\frac{4s^2}{L^2})}\int d^2\sigma\cos^2(\frac{n\pi\sigma}{R})\cos^2(\frac{k\pi\sigma}{R})\cos^2(\frac{r\pi\sigma}{R}) \\
&= \frac{1}{R^8L^2}\{(\sum_{m,n}\frac{n^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})(\sum_{r,s}\frac{r^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad +\frac{3}{2}[\sum_n n^4(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})](\sum_{r,s}\frac{r^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad +\frac{1}{4}\sum_{n,r}n^2r^2(n+r)^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{(n+r)^2}{R^2}+\frac{4l^2}{L^2}})(\sum_s\frac{1}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad +\frac{1}{4}\sum_{n>r}n^2r^2(n-r)^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{(n-r)^2}{R^2}+\frac{4l^2}{L^2}})(\sum_s\frac{1}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad +\frac{1}{4}\sum_{n<r}n^2r^2(r-n)^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{(r-n)^2}{R^2}+\frac{4l^2}{L^2}})(\sum_s\frac{1}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}})\} \\
&= \frac{\pi^3L}{8R^5}\{(\zeta(-1)E_2(q^{ann.}))^3+\frac{3}{2}\zeta(-1)E_2(q^{ann.})(\zeta(-2)+4H_{2,2}(q^{ann.}))+\frac{3}{4}F(q^{ann.})\},
\end{aligned}
\tag{A.30}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1' G \partial_1\partial_1' G \partial_0\partial_0' G \\
&= \frac{32}{R^7L^5}\sum_{n,k,r=1}^{\infty}\sum_{m,l,s=-\infty}^{\infty}\frac{n^2k^2s^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})(\frac{r^2}{R^2}+\frac{4s^2}{L^2})}\int d^2\sigma\cos^2(\frac{n\pi\sigma}{R})\cos^2(\frac{k\pi\sigma}{R})\sin^2(\frac{r\pi\sigma}{R}) \\
&= \frac{4}{R^6L^4}\{(\sum_{m,n}\frac{n^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})(\sum_{r,s}\frac{s^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad -[\sum_n n^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_s\frac{s^2}{\frac{n^2}{R^2}+\frac{4s^2}{L^2}})](\sum_l\frac{k^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}}) \\
&\quad +\frac{1}{2}[\sum_n n^4(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})](\sum_{r,s}\frac{s^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad -\frac{1}{4}\sum_{n,k}n^2k^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})(\sum_{r,s}\frac{s^2}{\frac{(n+k)^2}{R^2}+\frac{4s^2}{L^2}}) \\
&\quad -\frac{1}{4}\sum_{n,k}n^2k^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_l\frac{1}{\frac{n^2}{R^2}+\frac{4l^2}{L^2}})(\sum_{r,s}\frac{s^2}{\frac{(n-k)^2}{R^2}+\frac{4s^2}{L^2}})\} \\
&= \frac{\pi^3L}{8R^5}\{-(\zeta(-1)E_2(q^{ann.}))^3+\frac{1}{2}\zeta(-1)E_2(q^{ann.})(\zeta(-2)+4H_{2,2}(q^{ann.}))+\frac{3}{4}F(q^{ann.})\},
\end{aligned}
\tag{A.31}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_1\partial_1' G\partial_0\partial_0' G\partial_0\partial_0' G \\
&= \frac{128}{R^5L^7}\sum_{n,k,r=1}^{\infty}\sum_{m,l,s=-\infty}^{\infty}\frac{n^2l^2s^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})(\frac{r^2}{R^2}+\frac{4s^2}{L^2})}\int d^2\sigma\cos^2(\frac{n\pi\sigma}{R})\sin^2(\frac{k\pi\sigma}{R})\sin^2(\frac{r\pi\sigma}{R}) \\
&= \frac{16}{R^4L^6}\{(\sum_{m,n}\frac{n^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})(\sum_{r,s}\frac{s^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&-[\sum_n n^2(\sum_m\frac{1}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_s\frac{s^2}{\frac{n^2}{R^2}+\frac{4s^2}{L^2}})](\sum_l\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}}) \\
&+\frac{1}{2}[\sum_k(\sum_s\frac{s^2}{\frac{k^2}{R^2}+\frac{4s^2}{L^2}})(\sum_l\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})](\sum_{n,m}\frac{n^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}}) \\
&+\frac{1}{4}\sum_{r,k}(r+k)^2(\sum_s\frac{s^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}})(\sum_l\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})(\sum_m\frac{1}{\frac{(k+r)^2}{R^2}+\frac{4m^2}{L^2}}) \\
&+\frac{1}{4}\sum_{r,k}(r-k)^2(\sum_s\frac{s^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}})(\sum_l\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})(\sum_m\frac{1}{\frac{(k-r)^2}{R^2}+\frac{4m^2}{L^2}})\} \\
&= \frac{\pi^3L}{8R^5}\{(\zeta(-1)E_2(q^{ann.}))^3-\frac{1}{2}\zeta(-1)E_2(q^{ann.})(\zeta(-2)+4H_{2,2}(q^{ann.})+\frac{3}{4}F(q^{ann.})\},
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_0\partial_0' G\partial_0\partial_0' G\partial_0\partial_0' G \\
&= \frac{512}{R^3L^9}\sum_{n,k,r=1}^{\infty}\sum_{m,l,s=-\infty}^{\infty}\frac{m^2l^2s^2}{(\frac{n^2}{R^2}+\frac{4m^2}{L^2})(\frac{k^2}{R^2}+\frac{4l^2}{L^2})(\frac{r^2}{R^2}+\frac{4s^2}{L^2})}\int d^2\sigma\sin^2(\frac{n\pi\sigma}{R})\sin^2(\frac{k\pi\sigma}{R})\sin^2(\frac{r\pi\sigma}{R}) \\
&= \frac{64}{R^2L^8}\{(\sum_{m,n}\frac{m^2}{\frac{m^2}{R^2}+\frac{4m^2}{L^2}})(\sum_{k,l}\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})(\sum_{r,s}\frac{s^2}{\frac{r^2}{R^2}+\frac{4s^2}{L^2}}) \\
&+\frac{3}{2}[\sum_n(\sum_m\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_s\frac{s^2}{\frac{n^2}{R^2}+\frac{4s^2}{L^2}})](\sum_{k,l}\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}}) \\
&-\frac{1}{4}\sum_{n,k}(\sum_m\frac{m^2}{\frac{n^2}{R^2}+\frac{4m^2}{L^2}})(\sum_s\frac{s^2}{\frac{(n+k)^2}{R^2}+\frac{4s^2}{L^2}})(\sum_l\frac{l^2}{\frac{k^2}{R^2}+\frac{4l^2}{L^2}})\} \\
&= \frac{\pi^3L}{8R^5}\{-(\zeta(-1)E_2(q^{ann.}))^3-\frac{3}{2}\zeta(-1)E_2(q^{ann.})(\zeta(-2)+4H_{2,2}(q^{ann.}))+\frac{3}{4}F(q^{ann.})\} \quad .
\end{aligned}
\tag{A.33}
$$

A.4 The torus

The Green's function on the torus with periodicities L, R is given by

$$
G(\sigma_0,\sigma_1;\sigma_0',\sigma_1') = \frac{1}{4\pi^2RL}\sum_{(n,m)\neq(0,0)}\frac{e^{\frac{2\pi im}{L}(\sigma_0-\sigma_0')}e^{\frac{2\pi in}{R}(\sigma_1-\sigma_1')}}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} \quad . \tag{A.34}
$$

A.4.1 Partition function at $O(L^{-3})$

This was computed already in [9], and we reproduce it here for completeness. The diagrams

at this order are,

$$
\begin{aligned}
I_1^{tor.} &= \int d^2\sigma \partial_\alpha \partial^{\alpha'} G \partial_\beta \partial^{\beta'} G = \int d^2\sigma \{(\partial_0\partial_0' G + \partial_1\partial_1' G)(\partial_0\partial_0' G + \partial_1\partial_1' G)\} = \frac{1}{RL}, \\
I_2^{tor.} &= \int d^2\sigma \partial_\alpha \partial_{\beta'} G \partial^\alpha \partial^{\beta'} G = \int d^2\sigma \{\partial_0\partial_0' G \partial_0\partial_0' G + \partial_1\partial_1' G \partial_1\partial_1' G\} \\
&= \frac{\pi^2 L}{18R^3} E_2^2(q^{tor.}) - \frac{\pi}{3R^2} E_2(q^{tor.}) + \frac{1}{RL},
\end{aligned} \tag{A.35}
$$

and it is easy to check that they are invariant under $R \leftrightarrow L$ as they must be.

Below are the details of the computation :

$$
\begin{aligned}
\int d^2\sigma \partial_1\partial_1' G \partial_1\partial_1' G &= \frac{1}{R^5 L}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}} + \sum_{m\neq 0} 0)^2 \\
&= \frac{1}{R^5 L}(2\pi R L \zeta(-1) E_2(q^{tor.}))^2 = \frac{4\pi^2 L}{R^3}(\zeta(-1)E_2(q^{tor.}))^2,
\end{aligned} \tag{A.36}
$$

$$
\begin{aligned}
\int d^2\sigma \partial_0\partial_0' G \partial_1\partial_1' G &= \frac{1}{R^5 L}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}} + \sum_{m\neq 0} 0)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{m^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}} + \sum_{m\neq 0} L^2) \\
&= \frac{1}{R^3 L^3}(2\pi R L \zeta(-1) E_2(q^{tor.}))\left(-2\frac{\pi L^3}{R}\zeta(-1)E_2(q^{tor.}) + 2L^2\zeta(0)\right) \\
&= -\frac{4\pi^2 L}{R^3}(\zeta(-1)E_2(q^{tor.}))^2 + \frac{4\pi}{R^2}\zeta(0)\zeta(-1)E_2(q^{tor.}),
\end{aligned} \tag{A.37}
$$

$$
\begin{aligned}
\int d^2\sigma \partial_0\partial_0' G \partial_0\partial_0' G &= \frac{1}{RL^5}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{m^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}} + \sum_{m\neq 0} L^2)^2 \\
&= \frac{1}{RL^5}\left(-2\frac{\pi L^3}{R}\zeta(-1)E_2(q^{tor.}) + 2L^2\zeta(0)\right)^2 \\
&= \frac{4\pi^2 L}{R^3}(\zeta(-1)E_2(q^{tor.}))^2 - \frac{8\pi}{R^2}\zeta(0)\zeta(-1)E_2(q^{tor.}) + \frac{4}{RL}(\zeta(0))^2,
\end{aligned} \tag{A.38}
$$

A.4.2 Partition function at $O(L^{-5})$

At this order there are both 2-loop and 3-loop contributions (see figures 1 and 2) . At two loops there are two possible contractions, which both turn out to vanish:

$$
\begin{aligned}
I_3^{tor.} &= \int d^2\sigma \partial_\alpha \partial^{\alpha'} \partial_\beta \partial^{\beta'} G \partial_\gamma \partial^{\gamma'} G \\
&= \int d^2\sigma (\partial_0\partial_0'\partial_0\partial_0' G + \partial_1\partial_1'\partial_1\partial_1' G + 2\partial_1\partial_1'\partial_0\partial_0' G)(\partial_1\partial_1' G + \partial_0\partial_0' G) = 0,
\end{aligned} \tag{A.39}
$$

$$
I_4^{tor.} = \int d^2\sigma \partial_\alpha \partial_\beta \partial_{\gamma'} G \partial^\alpha \partial^\beta \partial^{\gamma'} G = 0. \tag{A.40}
$$

Below are the details of the computation :

$$\begin{aligned}\int d^2\sigma \partial_1\partial_1'\partial_1\partial_1' G \partial_1\partial_1' G &= -\int d^2\sigma \partial_1\partial_1'\partial_1\partial_1 G \partial_1\partial_1' G \\ &= \frac{4\pi^2}{R^7 L}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n^4}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0) \\ &= \frac{16\pi^4 L}{R^5}\zeta(-1)\zeta(-3)E_2(q^{tor.})E_4(q^{tor.}), \qquad \text{(A.41)}\end{aligned}$$

$$\begin{aligned}\int d^2\sigma \partial_1\partial_1'\partial_1\partial_1' G \partial_0\partial_0' G &= \frac{4\pi^2}{R^5 L^3}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n^4}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} L^2) \\ &= \frac{16\pi^3}{R^4}\left(-\frac{\pi L}{R}\zeta(-1)\zeta(-3)E_2(q^{tor.})E_4(q^{tor.})+\zeta(0)\zeta(-3)E_4(q^{tor.})\right), \qquad \text{(A.42)}\end{aligned}$$

$$\begin{aligned}\int d^2\sigma \partial_1\partial_1'\partial_0\partial_0' G \partial_1\partial_1' G &= -\int d^2\sigma \partial_1\partial_1\partial_0\partial_0' G \partial_1\partial_1' G \\ &= \frac{4\pi^2}{R^5 L^3}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^2 n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0) \\ &= -\frac{16\pi^4 L}{R^5}\zeta(-1)\zeta(-3)E_2(q^{tor.})E_4(q^{tor.}), \qquad \text{(A.43)}\end{aligned}$$

$$\begin{aligned}\int d^2\sigma \partial_1\partial_1'\partial_0\partial_0' G \partial_0\partial_0' G &= -\int d^2\sigma \partial_1\partial_1'\partial_0\partial_0 G \partial_0\partial_0' G \\ &= \frac{4\pi^2}{R^3 L^5}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^2 n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} L^2) \\ &= \frac{16\pi^3}{R^4}\left(\frac{\pi L}{R}\zeta(-1)\zeta(-3)E_2(q^{tor.})E_4(q^{tor.})-\zeta(0)\zeta(-3)E_4(q^{tor.})\right), \qquad \text{(A.44)}\end{aligned}$$

$$\begin{aligned}\int d^2\sigma \partial_0\partial_0'\partial_0\partial_0' G \partial_1\partial_1' G &= \frac{4\pi^2}{R L^7}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^4}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} m^2 L^2)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{n^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} 0) \\ &= \frac{16\pi^4 L}{R^5}\zeta(-1)\zeta(-3)E_2(q^{tor.})E_4(q^{tor.}), \qquad \text{(A.45)}\end{aligned}$$

$$\begin{aligned}\int d^2\sigma \partial_0\partial_0'\partial_0\partial_0' G \partial_0\partial_0' G &= -\int d^2\sigma \partial_0\partial_0\partial_0\partial_0' G \partial_0\partial_0' G \\ &= \frac{4\pi^2}{R L^7}(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^4}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} m^2 L^2)(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\frac{m^2}{\frac{m^2}{L^2}+\frac{n^2}{R^2}}+\sum_{m\neq 0} L^2) \\ &= -\frac{16\pi^4 L}{R^5}\zeta(-1)\zeta(-3)E_2(q^{tor.})E_4(q^{tor.})+\frac{16\pi^3}{R^4}\zeta(0)\zeta(-3)E_4(q^{tor.}). \qquad \text{(A.46)}\end{aligned}$$

At three loops there are 3 possible contractions :

$$
\begin{aligned}
I_6^{tor.} &= \int d^2\sigma \partial_\alpha \partial^{\alpha'} G \partial_\beta \partial^{\beta'} G \partial_\gamma \partial^{\gamma'} G = \int d^2\sigma \{ \partial_0 \partial_0' G \partial_0 \partial_0' G \partial_0 \partial_0' G + \partial_1 \partial_1' G \partial_1 \partial_1' G \partial_1 \partial_1' G \\
&\quad + 3\partial_0 \partial_0' G \partial_0 \partial_0' G \partial_1 \partial_1' G + 3\partial_1 \partial_1' G \partial_1 \partial_1' G \partial_0 \partial_0' G \} = \frac{8}{L^2R^2}\zeta(0)^3, \qquad \text{(A.47)} \\
I_7^{tor.} &= \int d^2\sigma \partial^\alpha \partial_\alpha' G \partial^\beta \partial_\gamma' G \partial_\beta' \partial^\gamma G = \int d^2\sigma \{ \partial_0 \partial_0' G \partial_0 \partial_0' G \partial_0 \partial_0' G + \partial_1 \partial_1' G \partial_1 \partial_1' G \partial_1 \partial_1' G \\
&\quad + \partial_0 \partial_0' G \partial_0 \partial_0' G \partial_1 \partial_1' G + \partial_1 \partial_1' G \partial_1 \partial_1' G \partial_0 \partial_0' G \} \\
&= -\frac{16\pi}{LR^3}\zeta(-1)E_2(q^{tor.})\zeta(0) + \frac{16\pi^2}{R^4}(\zeta(-1)E_2(q^{tor.}))^2\zeta(0) + \frac{8}{L^2R^2}\zeta(0)^3, \qquad \text{(A.48)} \\
I_8^{tor.} &= \int d^2\sigma \partial^\alpha \partial_\beta' G \partial^\beta \partial_\gamma' G \partial^\gamma \partial_{\alpha'} G = \int d^2\sigma \{ \partial_0 \partial_0' G \partial_0 \partial_0' G \partial_0 \partial_0' G + \partial_1 \partial_1' G \partial_1 \partial_1' G \partial_1 \partial_1' G \} \\
&= -\frac{24\pi}{LR^3}\zeta(-1)E_2(q^{tor.})\zeta(0) + \frac{24\pi^2}{R^4}(\zeta(-1)E_2(q^{tor.}))^2\zeta(0) + \frac{8}{L^2R^2}\zeta(0)^3 \quad . \qquad \text{(A.49)}
\end{aligned}
$$

Below are the details of the computation:

$$
\begin{aligned}
&\int d^2\sigma \partial_0 \partial_0' G \partial_0 \partial_0' G \partial_0 \partial_0' G = \frac{1}{R^2L^8}\left(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{m^2}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} + \sum_{m\neq 0} L^2 \right)^3 \\
&= -\frac{8\pi^3 L}{R^5}(\zeta(-1)E_2(q^{tor.}))^3 + \frac{24\pi^2}{R^4}(\zeta(-1)E_2(q^{tor.}))^2\zeta(0) - \frac{24\pi}{LR^3}\zeta(-1)E_2(q^{tor.})\zeta(0)^2 \\
&\quad + \frac{8}{L^2R^2}\zeta(0)^3, \qquad \text{(A.50)}
\end{aligned}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_0 \partial_0' G \partial_0 \partial_0' G \partial_1 \partial_1' G \\
&= \frac{1}{R^4L^6}\left(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{m^2}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} + \sum_{m\neq 0} L^2 \right)^2 \left(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{n^2}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} + \sum_{m\neq 0} 0 \right) \qquad \text{(A.51)} \\
&= \frac{8\pi^3 L}{R^5}(\zeta(-1)E_2(q^{tor.}))^3 - \frac{16\pi^2}{R^4}(\zeta(-1)E_2(q^{tor.}))^2\zeta(0) + \frac{8\pi}{LR^3}\zeta(-1)E_2(q^{tor.})\zeta(0)^2,
\end{aligned}
$$

$$
\begin{aligned}
&\int d^2\sigma \partial_0 \partial_0' G \partial_1 \partial_1' G \partial_1 \partial_1' G \\
&= \frac{1}{R^4L^6}\left(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{m^2}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} + \sum_{m\neq 0} L^2 \right) \left(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{n^2}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} + \sum_{m\neq 0} 0 \right)^2 \\
&= -\frac{8\pi^3 L}{R^5}(\zeta(-1)E_2(q^{tor.}))^3 + \frac{8\pi^2}{R^4}(\zeta(-1)E_2(q^{tor.}))^2\zeta(0), \qquad \text{(A.52)}
\end{aligned}
$$

$$
\begin{aligned}
\int d^2\sigma \partial_1 \partial_1' G \partial_1 \partial_1' G \partial_1 \partial_1' G &= \frac{1}{R^4L^6}\left(2\sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} \frac{n^2}{\frac{n^2}{R^2}+\frac{m^2}{L^2}} + \sum_{m\neq 0} 0 \right)^3 \qquad \text{(A.53)} \\
&= \frac{1}{R^8L^2}(2\pi RL\zeta(-1)E_2(q^{tor.}))^3 = \frac{8\pi^3 L}{R^5}(\zeta(-1)E_2(q^{tor.}))^3 \quad .
\end{aligned}
$$

A.5 $F(q)$: numerical evaluation

Due to technical difficulties in the evaluation of $F(q)$ as a finite sum of Eisenstein series, we study its modular properties using a numerical method. We are able to extract the coefficients in the series, $F(\tilde{q}) = \sum_n a_n(\frac{R}{L})^n + O(\tilde{q})$, which is a good approximation for $\frac{R}{L} \to \infty$. We first extract the divergent part which we compute with a zeta function regularization :

$$
\begin{aligned}
F(q) &= \sum_{n,r=1}^{\infty} (n^2 r + n r^2) \frac{1+q^n}{1-q^n} \frac{1+q^r}{1-q^r} \frac{1+q^{n+r}}{1-q^{n+r}} \\
&= \sum_{n,r=1}^{\infty} (n^2 r + n r^2) \left(1 + 2\frac{q^n}{1-q^n}\right) \left(1 + 2\frac{q^r}{1-q^r}\right) \left(1 + 2\frac{q^{n+r}}{1-q^{n+r}}\right) \\
&= 4 \left[\sum_{n=1}^{\infty} n^2 \left(\frac{q^n}{1-q^n}\right)\right] \left[-\frac{1}{12} + 2\sum_{r=1}^{\infty} r \left(\frac{q^r}{1-q^r}\right)\right] \\
&\quad +2 \sum_{n,r=1}^{\infty} (n^2 r + n r^2) \left(\frac{1+q^n}{1-q^n}\right) \left(\frac{1+q^r}{1-q^r}\right) \left(\frac{q^{n+r}}{1-q^{n+r}}\right).
\end{aligned}
\tag{A.54}
$$

We then sum F using this expression up to the $n, r = 1000$ term, and perform a fit for small $\tilde{q}$ of the form $F(\tilde{q}) = -a_5 \frac{\pi^4}{\log(q)^5} - a_4 \frac{\pi^2}{\log(q)^4} - a_3 \frac{1}{\log(q)^3} + O(\tilde{q})$. Our result (expressed as rational numbers times π's as expected) is

$$
\begin{aligned}
F(\tilde{q}) &= -\frac{\pi^4}{3\log(q)^5} - \frac{4\pi^2}{3\log(q)^4} - \frac{4}{3\log(q)^3} + O(\tilde{q}) \\
&= \frac{R^5}{3\pi L^5} - \frac{4R^4}{3\pi^2 L^4} + \frac{4R^3}{3\pi^3 L^3} + O(\tilde{q}).
\end{aligned}
\tag{A.55}
$$

B. Conventions for sections 4-6

B.1 General conventions

The coordinates we use are $Z^\mu = \{X^\alpha, X^i, Y^B, e^a\}$, where the indices are arranged in the following way (unless written otherwise) :

$$
\begin{aligned}
&\mu, \nu, \rho, \cdots = 0, \cdots, 9, \quad \alpha, \beta, \gamma, \cdots = 0, 1, \quad i, j, k, \cdots = 2, \cdots, D-1 \\
&\xi = 0, \cdots, D-1, \quad B = D, \cdots, D + N_B - 1, \quad a = D + N_B, \cdots, 9,
\end{aligned}
\tag{B.1}
$$

where N_B is the number of massive scalars, N_F is the number of massive fermions, N_B^0 is the number of massless transverse scalars and N_F^0 is the number of massless fermions.

We use the following notation to sum over the scalar fields,

$$
X \cdot X = \delta_{ij} X^i X^j, \quad Y \cdot Y = \delta_{ab} Y^a Y^b \quad . \tag{B.2}
$$

On the worldsheet we have the following metric and antisymmetric tensor :

$$
\eta_{\alpha\beta} = \eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.3}
$$

We define

$$k_1 \times k_2 \equiv \epsilon^{\alpha\beta} k_{1\alpha} k_{2\beta}. \tag{B.4}$$

We also use the lightcone coordinates $\tilde{\alpha} = (+,-)$, defined by the relation $\sigma^{\pm} = \sigma^0 \pm \sigma^1$. In these coordinates,

$$\begin{aligned} \eta_{\tilde{\alpha}\tilde{\beta}} &= \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad \eta^{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \quad \epsilon^{\tilde{\alpha}\tilde{\beta}} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \\ \frac{1}{4}k^2 &= \frac{1}{4}(k^\sigma k^\sigma - k^\tau k^\tau) = -k_+ k_-, \quad ik_\pm = \partial_\pm, \quad \partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1). \end{aligned} \tag{B.5}$$

B.2 Spinor conventions

Our spinor notation is almost identical to the one used in [35]. We choose the Majorana condition such that the fermions are real variables. This is consistent with choosing the conjugation operation to be $\bar{\theta} = \theta^T \Gamma^0$. We introduce the 10 space-time gamma matrices Γ_μ which satisfy $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$, and the chirality operator $\Gamma^{11} = \Gamma^0 \cdots \Gamma^9$. We use the notation $\Gamma_{\mu_1 \cdots \mu_n} = \frac{1}{n!}\Gamma_{[\mu_1} \cdots \Gamma_{\mu_n]}$ where the brackets indicate anti-symmetrization of the gamma matrices; e.g. $\Gamma_{01} = \frac{1}{2}(\Gamma_0\Gamma_1 - \Gamma_1\Gamma_0)$. The matrices are real and can be broken into blocks in the following way (using the metric (4.2)),

$$\Gamma_\alpha = \sqrt{2\pi\alpha'}\rho_\alpha \otimes I, \quad \Gamma_{i(B)} = \sqrt{2\pi\alpha'}\rho \otimes \gamma_{i(B)}, \quad \Gamma_a = \rho \otimes \gamma_a, \tag{B.6}$$

with the chirality operators,

$$\Gamma^{11} = \rho \otimes \gamma^c, \quad \rho = \rho_0\rho_1, \quad \gamma^c = \gamma^2 \cdots \gamma^9 \quad . \tag{B.7}$$

The worldsheet gamma matrices obey the following anti-commutation relations,

$$\{\rho_\alpha, \rho_\beta\} = 2\eta_{\alpha\beta}, \quad \{\rho_\alpha, \rho\} = 0 \quad . \tag{B.8}$$

We explicitly write the matrices we will use :

$$\begin{aligned} &\rho_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^+ = -2\rho_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad , \\ &\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^- = -2\rho_+ = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}, \quad \rho = \rho_0\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \end{aligned} \tag{B.9}$$

and the following relations :

$$\begin{aligned} \rho^+ &= \rho^0 + \rho^1, \quad \rho^- = \rho^0 - \rho^1, \\ \epsilon_{\alpha\beta}\rho^\alpha\partial^\beta &= \rho^-\partial_- - \rho^+\partial_+, \quad \eta_{\alpha\beta}\rho^\alpha\partial^\beta = \rho^+\partial_+ + \rho^-\partial_- \quad . \end{aligned} \tag{B.10}$$

The 8 dimensional gamma matrices γ_i obey flat space anti-commutation relations :

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad \{\gamma_i, \gamma^c\} = 0 \quad (i = 2, \cdots, 9) \quad . \tag{B.11}$$

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